Reflection with Absolute Generality

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Reflection in Pure Set Theory

The iterative conception of set

$V_0 = \emptyset$;

$V_{\alpha+1} = P(V_\alpha)$;

$V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$, where γ is a limit;

$V = \bigcup_{\alpha<Ord} V_\alpha$.

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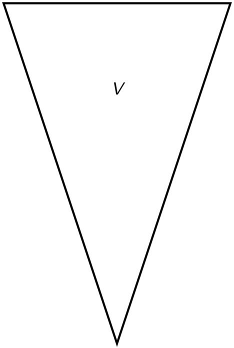

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Theorem (Lévy, Montague)

ZF $\vdash$ Lévy-Montague Reflection.

Second-order reflection

Bernays' Reflection

(RP_2) $\forall X[\varphi(X) \rightarrow \exists t(t \text{ is transitive} \wedge \varphi^t(X \cap t))]$, where φ is any formula in the language of class theory.

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An ω-Erdős cardinal is consistent with V=L, so RP_2 is a weak large cardinal axiom.

Set Theory with Urelements

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$V_0(A) = A$;
$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A)$;
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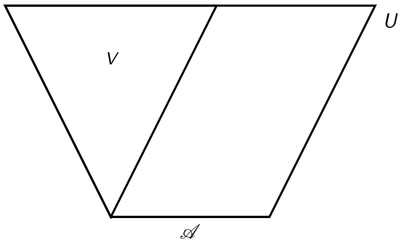

Question

How do reflection principles behave in urelement set theory?

First-Order Reflection

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Are RP and RP^- provable from "urelement set theory"? Are they equivalent?

$\mathsf{ZFU_R}$

Definition

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Definition

$ZFU_R = ZU + \text{Replacement}$.
$ZFCU_R = ZFU_R + AC$.
$ZF = ZFU_R + \forall x \neg \mathscr{A}(x)$.
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Note. The subscript R indicates that we are only working with Replacement.

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Start with a model $U \models \mathsf{ZFCU_R} + \mathscr{A} \sim \omega$. Let $U^{Fin} = \bigcup_{A \subseteq \mathscr{A}} V(A)$, where $A \subseteq \mathscr{A}$ is finite.

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Remark. This also shows that $ZFCU_R$ cannot prove the Collection Principle, i.e.,

$$\forall x \in w \exists y \varphi(x,y) \to \exists v \forall x \in w \exists y \in v\ \varphi(x,y).$$

Question

When will first-order reflection hold?

Plenitude and Tail

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(Tail) Every set of urelements has a tail.

A urelement-characterization of RP

Theorem

The following are equivalent over $ZFCU_R$.

- *RP*
- *RP^-*
- *Collection*
- *Plenitude* $\vee$ *Tail*

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This provides a characterization of first-order reflection in terms of urelements.

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Theorem

- $\text{ZFU}_R + \textit{Plenitude} \nvdash \textit{RP}$ (in fact, *Collection*);
- $\text{ZFU}_R + \textit{RP} \nvdash (\textit{Plenitude} \vee \textit{Tail})$.

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Open Questions

- $\mathrm{ZFU}_R + \mathit{Collection} \vdash \mathit{RP}^-$?
- $\mathrm{ZFU}_R + \mathit{RP}^- \vdash \mathit{RP}$?
- $\mathrm{ZFU}_R + \mathit{RP}^- \vdash \mathit{Collection}$?
- ...

Urelement class theory

The language of *urelement class theory* is two-sorted: the first-order variables $w, x, y, z, \dots$ quantify over sets and urelements, and the second-order variables $X, Y, R, F, \dots$ quantify over classes.

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$$\text{(Collection) } \forall x \in w\ \exists y R(x, y) \rightarrow \exists v \forall x \in w \exists y \in v\ R(x, y).$$

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(Collection) $\forall x \in w\, \exists y R(x,y) \to \exists v \forall x \in w \exists y \in v\, R(x,y)$.

(RP) For every X_1, ..., X_n, there is a transitive set t such that for every $x_1, \ldots, x_m \in t$,

$$\varphi(X_1, \ldots, X_n, x_1, \ldots, x_m) \leftrightarrow \varphi^t(X_1 \cap t, \ldots, X_n \cap t, x_1, \ldots, x_m),$$

where φ contains only first-order quantifiers.

Urelement class theories

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$KMcU_R$ = KMU_R + AC.
GBCU= GBU_R + Global Well-Ordering (GWO)
KMCU = KMU_R + GWO.

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Theorem (Felgner)

Over KMcU_R,

- *Global Choice* $\nrightarrow$ *Global Well-Ordering;*
- *Global Well-Ordering* $\nrightarrow$ *Limitation of Size.*

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Open Question

KMcU_R + Collection + Global Choice $\vdash$ RP?

RP_2 with Urelements

Recall Bernays' second-order reflection principle.

$$(RP_2)\ \forall X[\varphi(X) \to \exists t(t \text{ is transitive} \land \varphi^t(X \cap t))],$$

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Question

Can urelements affect the strength of RP_2?

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Question

Is $V < \mathscr{A}$ consistent with RP_2?

The $U_{\kappa,A}$-hierarchy

Definition

Let κ be an infinite cardinal and $A \subseteq \mathscr{A}$.

$$U_{\kappa,A} = \bigcup_{B \in P_\kappa(A)} V_\kappa(B),$$

where $P_\kappa(A) = \{x \subseteq A : x < \kappa\}$.

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Zermelo's Quasi-Categoricity Theorem.

A full second-order model $\mathscr{M}$ satisfies ZFC_2 iff $\mathscr{M}$ is isomorphic to some V_κ, where κ is inaccessible.

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A full second-order model $\mathscr{M}$ satisfies ZFC_2 iff $\mathscr{M}$ is isomorphic to some V_κ, where κ is inaccessible.

$U_{\kappa,A}$ is a natural generalization of V_κ in the context of urelement set theory.

Theorem

Let M be a transitive set. The following are equivalent.

- $\langle M, P(M)\rangle \models KMCU$.
- $M = U_{\kappa,A}$ *for some inaccessible cardinal* κ *and* $A \subseteq \mathscr{A}$.

Moreover, $U_{\kappa,A} \models V < \mathscr{A}$ *if* $\kappa < A$.

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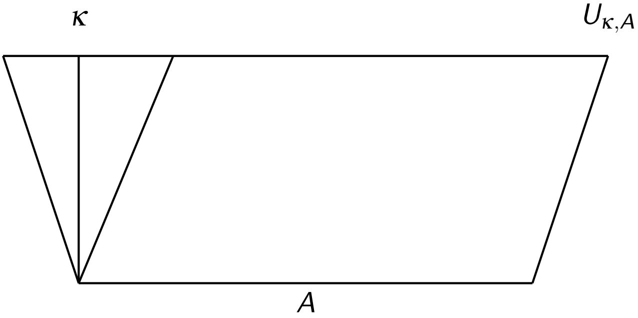

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Assume the consistency of a κ^+-supercompact cardinal κ. There is a $U_{\kappa,A}$ model such that $U_{\kappa,A} \models RP_2 \wedge V < \mathscr{A}$.

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Question

What is the strength of $\mathrm{RP}_2 + V < \mathscr{A}$?

Definition (Hamkins, Y.)

The Abundant Atom Axiom (AAA) $=_{\mathrm{df}}$

- $V < \mathscr{A}$;
- for every small class B (i.e, $B < \mathscr{A}$) there is a small $D \subseteq I \times B$ such that every subclass of B is D_i for some $i \in I$ ("$\mathscr{A}$ **strong limit**");
- if I is small and $D \subseteq I \times B$ is such that D_i is small for each $i \in \mathscr{I}$, then D itself is small ("$\mathscr{A}$ **regular**").

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Proposition (Hamkins, Y.)

- If $\kappa < \kappa'$ are both inaccessible and $|A| = \kappa'$, then $U_{\kappa,A} \models \mathsf{AAA}$;
- if κ is $< \lambda$-supercompact for some inaccessible $\lambda > \kappa$, then there is a model of $U_{\kappa,A} \models \mathsf{RP}_2 + \mathsf{AAA}$.

Theorem (Hamkins, Y.)

KMCU+ RP_2 + AAA interprets KM + a supercompact cardinal. Moreover, it implies the existence of a proper class of measurable cardinals, and more.

Thank You!