Full and Mixed (Boolean-Valued) Models

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Introduction

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- Logical connectives and quantifiers are interpreted as algebratic operations on $\mathbb{2}$: conjunction as Boolean meet, negation as Boolean complementation, etc.
- A natural way to generalize the classical models: use an arbitrary complete Boolean algebra as the range of truth values.

Boolean-Valued Models

Boolean Algebra

A *Boolean algebra* is a partial order $(B, \leqslant)$ s.t. for any $p, q, r \in B$,

1. p, q have a *lub* $(p \vee q)$ and a *glb* $(p \wedge q)$.
2. there is a greatest element 1 and a least element 0.
3. p has a complement $(-p)$ s.t. $p \vee -p = 1$ and $p \wedge -p = 0$.
4. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$,
   $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

A complete Boolean algebra is a Boolean algebra where each subset of B has a least upper bound with respect to $\leqslant$.

Boolean-Valued Models

Let $\mathscr{L}$ be a first-order language. Let $\mathbb{B}$ be a complete Boolean algebra.

Boolean Algebra

A $\mathbb{B}$-valued model $\mathfrak{M}$ for $\mathscr{L}$ consists of:

1. A universe M of elements.
2. Interpretation of any constant C: $[\![C]\!] \in M$.
3. Interpretation of $=$: a function Ω from M^2 to $\mathbb{B}$, and interpretation of any n-ary relation P: a function h_P from M^n to $\mathbb{B}$, s.t. for any $x_1, ..., x_n, y_1, ..., y_n \in D$,
   - $\Omega(x_1, x_1) = 1$.
   - $\Omega(x_1, x_2) = \Omega(x_2, x_1)$.
   - $\Omega(x_1, x_2) \wedge \Omega(x_2, x_3) \leqslant \Omega(x_1, x_3)$.
   - $h_P(x_1, ..., x_n) \wedge \prod_{1 \leqslant i \leqslant n} \Omega(x_i, y_i) \leqslant h_P(y_1, ..., y_n)$.

Boolean-Valued Models

We define satisfaction as follows. Let $t_1, \ldots t_n$ be terms, $\mathbf{x}$ be a variable assignment:

1. $[\![C]\!][\mathbf{x}] = [\![C]\!]$.
2. $[\![v_i]\!][\mathbf{x}] = x_i$, where $x_i = \mathbf{x}(v_i)$.
3. $[\![t_1 = t_2]\!][\mathbf{x}] = \Omega([\![t_1]\!], [\![t_2]\!])$.
4. $[\![P(t_1, \ldots, t_n)]\!][\mathbf{x}] = h_P([\![t_1]\!], \ldots, [\![t_n]\!])$.
5. $[\![\neg\phi]\!][\mathbf{x}] = \neg[\![\phi]\!][\mathbf{x}]$.
6. $[\![\phi \wedge \psi]\!][\mathbf{x}] = [\![\phi]\!][\mathbf{x}] \wedge [\![\psi]\!][\mathbf{x}]$.
7. $[\![\forall v \phi]\!][\mathbf{x}] = \bigwedge_{x \in M}[\![\phi]\!][\mathbf{x}(v/x)]$.

First-order Boolean-valued models are sound and complete with respect to classical first-order logic.

Definition

Let T be a theory and ϕ be a sentence in $\mathscr{L}$. $T \models_B \phi$ iff for any Boolean-valued model $\mathfrak{M}$ of $\mathscr{L}$, if $\mathfrak{M} \models T$, then $\mathfrak{M} \models \phi$.

Soundness and Completeness (Rasiowa, Sikorski)

$T \models_B \phi$ iff $T \vdash \phi$.

Full Models

Full models have "witnesses" for existential formulas:

Full Models

A $\mathbb{B}$-valued model $\mathfrak{M}$ is *full* iff for any formula $\phi(v, \overline{v})$ in $\mathscr{L}$, any $\overline{x} \subseteq M$, there is some $x \in M$ such that $[\![\exists v \phi(v, \overline{x})]\!] = [\![\phi(x, \overline{x})]\!]$.

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Quotient Models

Let $\mathfrak{M}$ be a $\mathbb{B}$-valued model and h be a homomorphism from $\mathbb{B}$ to $\mathbb{B}'$. The quotient model $\mathfrak{M}/h$ is the $\mathbb{B}'$-valued model defined as follows:

1. The universe $M/h = \{[x]_h \mid x \in M\}$, where $x_1 \equiv_h x_2$ iff $h(\llbracket x_1 = x_2 \rrbracket^{\mathfrak{M}}) = 1_{\mathbb{B}'}$.
2. For any n-ary predicate P, $\llbracket P(\overline{[x]_h}) \rrbracket^{\mathfrak{M}/\mathfrak{h}} = h(\llbracket P(\overline{x}) \rrbracket^{\mathfrak{M}})$.
3. For any constant C, C refers to $[\llbracket C \rrbracket^{\mathfrak{M}}]_h$.

Łos' Theorem

Quotient Models

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Generalized Łos' Theorem (Folklore)

Let $\mathfrak{M}$ be a full $\mathbb{B}$-valued model and h be a homomorphism from $\mathbb{B}$ to $\mathbb{B}'$. For any formula $\phi(\overline{v})$, any $\overline{x} \in M$, $\llbracket \phi(\overline{[x]_h}) \rrbracket^{\mathfrak{M}/h} = \llbracket \phi(\overline{x}) \rrbracket^{\mathfrak{M}}$.

Application: Set-theoretic Forcing

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To prove ϕ is independent of ZFC: find a $\mathbb{B}$ such that $0 < [\![\phi]\!]^{V^{\mathbb{B}}} < 1$. Find ultrafilters D^+, D^- on $\mathbb{B}$ such that $[\![\phi]\!] \in D^+$ and $\neg[\![\phi]\!] \in D^-$.

Downward Löwenheim-Skolem

Elementary Submodel

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two $\mathbb{B}$-valued models of $\mathscr{L}$. $\mathfrak{M}_1$ is an *elementary submodel* of $\mathfrak{M}_2$ iff for any ϕ of $\mathscr{L}$, any $x_1, ..., x_n \in M_1$, $[\![\phi(x_1, ..., x_n)]\!]^{\mathfrak{M}_1} = [\![\phi(x_1, ..., x_n)]\!]^{\mathfrak{M}_2}$.

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Strong Downward Löwenheim-Skolem

Let $\mathfrak{M}$ be an infinite and full $\mathbb{B}$-valued model of $\mathscr{L}$. Then $\mathfrak{M}$ has an elementary submodel of size $|\mathscr{L}|$.

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Let $\mathfrak{M}$ be an infinite and full $\mathbb{B}$-valued model of $\mathscr{L}$. Then $\mathfrak{M}$ has an elementary submodel of size $|\mathscr{L}|$.

Just as the Łos' Theorem, this only holds on full models:

Observation

There exists a Boolean-valued model that does not have a countable elementary submodel.

Boolean Valuations

Rasiowa and Sikorski's theorems only concern sentences with truth value 1 in Boolean-valued models.

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Boolean-Valuation

A *Boolean-valuation* $S^{\mathbb{B}}$ in $\mathscr{L}$ is a set of pairs of the form $\langle \phi, p \rangle$ such that ϕ is a sentence of $\mathscr{L}$ and p is an element of $\mathbb{B}$.

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Definition

A $\mathbb{B}$-valued model $\mathfrak{M}$ is a model of, or satisfies $S^{\mathbb{B}}$ iff for any sentence $\phi \in \mathscr{L}$, for any $p \in \mathbb{B}$, if $\langle \phi, p \rangle \in S^{\mathbb{B}}$, then $[\![\phi]\!]^{\mathfrak{M}} = p$.

Boolean Valuations

What does it mean for a Boolean-valuation to be *consistent*?

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Consistency

Let $S^{\mathbb{B}}$ be a Boolean-valuation of $\mathscr{L}$. Let $h : \mathbb{B} \to 2$ be a homomorphism. $S^{\mathbb{B}}_h$ is the following set of sentences: for any $\phi \in \mathscr{L}$, any $p \in B$,

1. If $\langle \phi, p \rangle \in S^{\mathbb{B}}$ and $h(p) = 1$, then $\phi \in S^{\mathbb{B}}_h$.
2. If $\langle \phi, p \rangle \in S^{\mathbb{B}}$ and $h(p) = 0$, then $\neg\phi \in S^{\mathbb{B}}_h$.
3. Nothing else is in $S^{\mathbb{B}}_h$.

A Boolean-valuation $S^{\mathbb{B}}$ is *consistent* if and only if for any homomorphism $h : \mathbb{B} \to 2$, $S^{\mathbb{B}}_h$ is a consistent theory.

Boolean Valuations

Lemma

Let $S^{\mathbb{B}}$ be a consistent Boolean-valuation. For any sentence $\phi \in \mathscr{L}$, for some $p \in \mathbb{B}$, $S^{\mathbb{B}} \cup \{\phi, p\}$ is consistent.

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Completeness

Let $S^{\mathbb{B}}$ be a consistent Boolean-valuation of $\mathscr{L}$. Then $S^{\mathbb{B}}$ has a full $\mathbb{B}$-valued model of size $\leqslant |\mathscr{L}|$.

Weak Downward Löwenheim-Skolem

If $S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model, then it has a full $\mathbb{B}$-valued model of size $\leqslant |\mathscr{L}|$.

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Soundness

If $S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model, then $S^{\mathbb{B}}$ is consistent.

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If $S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model, then it has a full $\mathbb{B}$-valued model of size $\leqslant |\mathscr{L}|$.

Soundness

If $S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model, then $S^{\mathbb{B}}$ is consistent.

Compactness

$S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model iff every finite subset of $S^{\mathbb{B}}$ has a $\mathbb{B}$-valued model.

Elementary Submodel and Elementary Diagram

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Elementary Diagram

Let $\mathfrak{M}$ be a $\mathbb{B}$-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{M}} = \mathscr{L} \cup \{C_x \mid x \in M\}$, where $\{C_x \mid x \in M\}$ is a new set of constants, one for each $x \in M$. The canonical expansion of $\mathfrak{M}$ to $\mathscr{L}_{\mathfrak{M}}$ is the model $\mathfrak{M}^*$ where for all $x \in M$, $[\![C_x]\!]^{\mathfrak{M}^*} = x$.

The *elementary diagram* of $\mathfrak{M}$ is the $\mathfrak{B}$-valuation $S^{\mathbb{B}}_{\mathfrak{M}}$ which consists of and only of all the pairs of the form $\langle \phi, [\![\phi]\!]^{\mathfrak{M}^*} \rangle$ where ϕ is a sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{M}^*}$ is the value of ϕ in $\mathfrak{M}^*$.

Full Elementary Extension

Equivalence

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two $\mathbb{B}$-valued models. TFAE:

1. $\mathfrak{M}_1$ is isomorphic to an elementary submodel of $\mathfrak{M}_2$.
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Using this equivalence and the Completeness theorem on Boolean-valuations:

Corollary

Every Boolean-valued model has a full elementary extension.

Application: Forcing with Urelements

The existing method (Blass, Ščedrov) of constructing Scott-Solovay style models of set theory with urelements gives us Boolean-valued models that are not full.

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This is a problem, as the method for obtaining relative consistency results via the Łos' Theorem requires full models.

But the result on the previous slide solves the problem, as we can use Łos' Theorem on their elementary extensions.

Mixed Models

A $\mathbb{B}$-valued model $\mathfrak{M}$ is *mixed* iff for any antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ and any $\{x_i \mid i \in I\} \subseteq M$, there is some $x \in M$ such that for any $i \in I$, $p_i \leqslant [\![x = x_i]\!]$.

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What is the connection between mixed models and full models?

Highly Full Models

A $\mathbb{B}$-valued model $\mathfrak{M}$ is *highly full* iff for any language $\mathscr{L}'$ that expands $\mathscr{L}$, any expansion $\mathfrak{M}'$ of $\mathfrak{M}$ to $\mathscr{L}'$, $\mathfrak{M}'$ is full.

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Theorem

$\mathfrak{M}$ is mixed iff for $\mathfrak{M}$ is highly full.

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Theorem

Let $\mathfrak{M}$ be a mixed $\mathbb{B}$-valued model of a countable language $\mathscr{L}$. Let $h : \mathbb{B} \to 2$ be a countably incomplete homomorphism. Then the quotient model $\mathfrak{M}/h$ is ω_1-saturated.

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Theorem

Let $\mathfrak{M}$ be a mixed $\mathbb{B}$-valued model and $h : \mathbb{B} \to \mathbb{C}$ be a homomorphism. Then, for any Σ^1_1 formula ϕ, any $x_1, ..., x_n \in M, h(\llbracket \phi(x_1, ..., x_n) \rrbracket^{\mathfrak{M}}) \leqslant \llbracket \phi([x_1]_h, ..., [x_n]_h) \rrbracket^{\mathfrak{M}/h}$.

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Conjecture

Every Boolean-valued model has a mixed elementary extension.

Thank you for listening! :)