Strong Kleene Supervaluation and Theories of Naive Truth

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Bristol Logic Meeting

European Research Council
Established by the European Commission
ERC-2018-STG 803684 TRUST

University of BRISTOL

Setting Expectations

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- We unashamedly call exercises "Propositions", "Theorems", or "Lemmas"…

Introduction

Tarski and Truth

- Convention T
- Undefinability Theorem
- Defining truth in an “essentially stronger metalanguage”.
- Typing and hierarchies
- Self-applicability?

Kripke and Truth

- Partial logics and positive inductive definitions
- Modified convention T

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Classical supervaluation

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Many-valued logic

- Compositional truth-conditions;
- Conditionals/Conditional reasoning?

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Naivity

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Denoting Conditionals

Focus on the determiner Every

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Denoting Conditionals

Focus on the determiner Every

- $\forall x\varphi := \mathrm{Every}_x(\top, \varphi)$;
- $\varphi \rightarrow \psi := \mathrm{Every}_x(\varphi, \psi)$ with $x \notin FV(\varphi \wedge \psi)$.

Truth, Conditionals, and Curry

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- Curry's paradox main obstacle for conditionals/RQ in non-classical truth theories.
- Orthodox TC: κ is true iff κ is not in the interpretation of the truth predicate.
- No naive truth models with orthodox TCs

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- Transparency, structural rules, and deduction theorem
- Transparency + MP + $\to$-contraction + $\to$-reflexivity

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- **Logicality of** $\rightarrow$: Conditional defined relative to a model class also containing non-naive truth models.
- **Logicality of truth**: Conditional defined relative to naive truth models only; loss of crucial logical properties of $\rightarrow$.

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- We opt for the logicality of $\rightarrow$.

Conditionals and Truth in Partial Logics

Aim

Construct a naive truth model with a "logical" conditional.

- Conditional interpreted as truth preservation
- Not local: the naive truth model needs to "see" non-naive models
- stability under semantic precisifications/local domain extensions ("Monotonicity")
- Form of intuitionistic conditional

Strong Kleene Supervaluation

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Supervaluation structure $\mathfrak{M}$

A tuple (D, X, H) such that $D \neq \emptyset$ and

- X is a set of partial (strong Kleene) interpretations such that for all $I, J \in X$ and all closed terms t
  - $J(t) = I(t)$
- $H \subseteq X \times X$ such that
  - H is transitive
  - if $(I, J) \in H$, then $I \leq J$.

Truth relative to an Interpretation

Let $J \in X$ and $\|\chi\|_x^{J,\beta} = \{d \in D \,|\, \mathfrak{M}, J \Vdash \varphi[\beta(x : d)]\}$.:

$$\mathfrak{M}, J \Vdash \text{Every}_x(\varphi, \psi)[\beta] \quad \text{iff } \forall J'((J, J') \in H \Rightarrow \|\varphi\|_x^{J',\beta} \subseteq \|\psi\|_x^{J',\beta})$$

$$\mathfrak{M}, J \Vdash \neg\text{Every}_x(\varphi, \psi)[\beta] \quad \text{iff } \|\varphi\|_x^{J,\beta} \cap \|\neg\psi\|_x^{J,\beta}) \neq \emptyset$$

- strong Kleene truth for remaining clauses.

Taking Stock

Non-classical supervaluation

Constant domain intuitionistic Kripke frames with inclusion negation.

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Logic

- Corresponds to Nelson logic (N3);
  - $\to :\approx$ sk-sequent arrow in object lang.
- Disjunction and existence property;
- Some flexibility:
  - Use fde-style semantics: N4, Hype (QN*),…
  - Strengthening of tc for Every to allow for contraposition

Truth

Interpreting Truth

Expand supervaluation structure $\mathfrak{M} = (D, X, H)$ for $\mathcal{L}$ to an supervaluation structure for $\mathcal{L}_{\mathrm{T}}$

Assumptions

- $\mathcal{L}$ extends the language of some syntax theory $\mathcal{L}_S$, e.g., the language of arithmetic;
- $\mathcal{L}$ contains names of all elements of D;
- for all $\varphi \in \mathcal{L}_S$; $J, J' \in X$ and assignments β.
  - $\mathfrak{M}, J \Vdash \varphi[\beta]$ iff $\mathfrak{M}, J' \Vdash \varphi[\beta]$
  - $\mathfrak{M}, J \Vdash \varphi \vee \neg\varphi[\beta]$

Valuation on $\mathfrak{M}$

Function that assigns an interpretation to the truth predicate relative to a world and an interpretation:

- $f : X \rightarrow \mathcal{P}(\mathrm{Sent})$

Admissible Valuations

Not all valuations are equally good. A valuation f is admissible on $\mathfrak{M} = (D, X, H)$ iff

- f is consistent, i.e., if for all $J \in X$ and $\varphi \in \mathcal{L}_{\mathrm{T}}$:

$$\varphi \notin f(J) \text{ or } \neg\varphi \notin f(J);$$

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- for all $J \in X$ and $\varphi \in \mathcal{L}$:

$$\text{if } \varphi \in f(J), \text{ then } (\mathfrak{M}, J) \Vdash \varphi;$$

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- for all $J, J' \in X$, if $(J, J') \in H$, then $f(J) \subseteq f(J')$.

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Truth Interpretation

Let $J \in X$ and f an admissible valuation, then J_f is a called a truth-interpretation for the language $\mathcal{L}_{\mathrm{T}}$:

$$J_f(P) := \begin{cases} f(J), & \text{if } P \doteq \mathrm{T}; \\ J(P), & \text{otherwise.} \end{cases}$$

Admissibility Condition

Ordering

Let f, g be valuations of $\mathfrak{M}$. Then $f \leq g$ iff $f(w, J) \subseteq g(J)$, for all $J \in X$.

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$$\text{if } g \in \Phi(f), \text{ then } f \leq g.$$

- Φ yields the admissible precisifications of an valuations f
- Φ induces an ordering on $\text{Val}_{\mathfrak{M}}^{\text{Adm}}$: $f \leq_\Phi g :\leftrightarrow g \in \Phi(f)$.

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Further Assumptions:

- $\leq_\Phi$ is transitive
- if $f \leq g$, then $\Phi(g) \subseteq \Phi(f)$.

Truth Structure

Let $\mathfrak{M} = (D, X, H)$ be a supervaluation structure and $Y \subseteq \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}$. Then the tupel $(D, X \times Y, H_\Phi)$ is called a **truth structure** iff for all $I, J \in X$ and $f, g \in Y$:

$$(I_f, J_g) \in H_\Phi :\leftrightarrow (I, J) \in H \,\&\, f \leq_\Phi g.$$

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Grounded Truth Structure

Let $\mathfrak{M}_\mathrm{T} = (D, X \times Y, H_\Phi)$ be a truth structure. If there is an $f \in Y$ such that $Y \cap \Phi(f) \neq \emptyset$ and $f \leq g$ for all $g \in Y$, then $\mathfrak{M}_\mathrm{T}$ is called a **grounded truth structure**. A set Y_f with minimal element f is called a grounded truth set.

Truth Structures and Kripkean Truth

- Truth structure give an interpretation of $\mathcal{L}_{\mathrm{T}}$;
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Aim

Find a grounded truth structure $\mathfrak{M}_{\mathrm{T}}$ with minimal $f \in Y$ such that for all $J \in X$, $w \in W$ and $\varphi \in \mathcal{L}_{\mathrm{T}}$:

$$\mathfrak{M}_{\mathrm{T}}, J_f \Vdash \mathrm{T}\ulcorner\varphi\urcorner \text{ iff } \mathfrak{M}_{\mathrm{T}}, J_f \Vdash \varphi.$$

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- Transparency is out of reach!

Fixed Points

Definition (Compactness of Φ)

Set $\Phi(X) = \{\Phi(f) \mid f \in X\}$*.* Φ *is compact on* $\mathsf{Val}_{\mathfrak{M}}^{\mathsf{Adm}}$ *iff for all* $X \subseteq \mathsf{Val}_{\mathfrak{M}}^{\mathsf{Adm}}$*: if* $\Phi(f_1) \cap \ldots \cap \Phi(f_n) \neq \emptyset$ *for all* $n \in \omega$ *and* $f_1, \ldots f_n \in X$*, then* $\bigcap \Phi(X) \neq \emptyset$*.*

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Proposition

Let $\mathfrak{M} = (D, X, H)$ *be a supervaluation structure and* Φ *compact on* $\mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}$*. Then there exists a grounded truth set* Y_f *and admissible valuation function* f *such that for all* $\varphi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{T}}}$

$$(D, X \times Y_f, H_\Phi), J_f \Vdash \varphi \text{ iff } (D, X \times Y_f, H_\Phi), J_f \Vdash \mathrm{T}\ulcorner\varphi\urcorner$$

for all $J \in X$.

Some more specifics

Let $\mathrm{Adm}_{\mathfrak{M}}$ be the set of grounded truth sets. Define two operations:

- $\theta^{\Phi}_{\mathfrak{M}} : \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}} \times \mathrm{Adm}_{\mathfrak{M}} \to \mathrm{Val}_{\mathfrak{M}}$ such that for all $f \in Y_f \in \mathrm{Adm}_{\mathfrak{M}}$ and $J \in X$:

$$[\theta^{\Phi}_{\mathfrak{M}}(f, Y_f)](J) := \{\varphi \mid (F, X \times Y_f, H_\Phi), J_f \Vdash \varphi\}$$

- $\Theta^{\Phi}_{\mathfrak{M}} : \mathrm{Adm}_{\mathfrak{M}} \to \mathcal{P}(\mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}})$ such that for all $Y_f \in \mathrm{Adm}_{\mathfrak{M}}$:

$$\Theta^{\Phi}_{\mathfrak{M}}(Y_f) = \{g \in Y_f \mid \theta^{\Phi}_{\mathfrak{M}}(Y_f, f) \leq g\}.$$

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Observation

Let $f \in Y_f \in \mathrm{Adm}_{\mathfrak{M}}$. Then

$$\theta(Y_f, f) = f \text{ iff } \Theta(Y_f) = Y_f.$$

Iterating Θ

$$\Theta^{\alpha}(Y_f) := \begin{cases} Y_f, & \text{if } \alpha = 0; \\ \Theta(\Theta^{\beta}(Y_f)), & \text{if } \alpha = \beta + 1 \text{ and } \Theta^{\beta}(Y_f) \in \mathrm{Adm}_{\mathfrak{M}}; \\ \emptyset, & \text{if } \alpha = \beta + 1 \text{ and } \Theta^{\beta}(Y_f) \notin \mathrm{Adm}_{\mathfrak{M}}; \\ \bigcap_{\beta \leq \alpha}(\Theta^{\beta}(Y_f), & \text{if } \alpha \text{ is limit.} \end{cases}$$

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‘Naive’ Fixed Point Property

$$\Phi_{\mathrm{Nve}}(f) := \begin{cases} \emptyset, & \text{if } f \notin \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}; \\ \{g \in \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}} \mid f \leq g \,\&\, g \text{ is (N3)-naive}\}, & \text{otherwise.} \end{cases}$$

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Proposition (Φ_{Nve}-fixed points)

Let $\mathfrak{M} = (D, X, H)$ be a supervaluation structure. The there exists a grounded truth set Y_f

$$\theta(Y_f, f) = f \text{ and } \Theta(Y_f) = Y_f$$

with admissibility condition Φ_{Nve}.

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- Naive valuation functions and transparency

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Deduction Theorem

Let J_f a fixed-point and $\mathfrak{M}_{J_f}$ the J_f generated substructure of $\mathfrak{M}$. Then

$$\Gamma, \varphi \vDash_{\mathfrak{M}_{J_f}} \psi \text{ iff } \Gamma \vDash_{\mathfrak{M}_{J_f}} \varphi \rightarrow \psi$$

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θ-compactness

If $\Phi(\theta^{\alpha}(f, Y_f)) \cap Y_f \neq \emptyset$ for $\alpha \leq \xi$, then $\Phi(\theta^{\xi}(f, Y_f)) \cap Y_f \neq \emptyset$.

- $\Phi(\theta^{\xi}(f, Y_f))$ is not ω-inconsistent.
- Consistent in ω-logic?

N3-saturation?

ω-consistency

There are fixed points for

$$\Phi_{\omega-\mathrm{Nve}}(f) := \begin{cases} \emptyset, & \text{if } f \notin \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}}; \\ \{g \in \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}} \mid f \leq g \,\&\, g \text{ is naive a. } \omega \text{ cons.}\}, & \text{else.} \end{cases}$$

Question

Can we find fixed for Φ selecting

- N3-saturated precisifications/sets
- N3-saturated and naive precisifications/sets

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Lemma

Let $\mathfrak{M}_\mathrm{T} = (D, X \times Y_f, H_{\Phi_{\mathrm{Nve}}}$ *with* $f \in \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}}$ *and* $Y_f = \{g \in \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}} \mid f \leq g\}$*. Then,* $f \leq \theta_{\mathfrak{M}_\mathrm{T}}(f, Y_f)$ *implies that* $[\theta_{\mathfrak{M}_\mathrm{T}}(f, Y_f)](J)$ *is a* Π^1_1*-hard for all* $J \in X$*.*

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Corollary

Let $\mathfrak{M} = (D, J, \{< J, J >\}) = \mathcal{N}$. *Then there exists no* $\Sigma \subseteq \mathcal{L}_{\mathrm{T}}$ *such that*

$$\theta^{\Phi_{\mathrm{Nve}}}_{\mathfrak{M}}(f, Y_f) = f \text{ iff } (\mathcal{N}, f(J)) \Vdash \Sigma.$$

Outlook

- Modal strong Kleene supervaluation: modality and natural language conditionals
- First-order approaches
  - External and internal axiomatizations
- Generalized quantifiers
- Intuitionistic supervaluation