Some questions in second order set theory

Philipp Schlicht, University of Bristol

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Second order arithmetic

The consistency strength of many statements about continuous functions on the reals, countable graphs and countable algebraic structures is at the level of subsystems of second order arithmetic Z_2.

Definition

Z_2 describes the 2-sorted structure $(\mathbb{N}, P(\mathbb{N}))$ in the language $\mathcal{L} = \{0, S, +, \cdot, =, <, \in\}$:

- Basic axioms about addition and multiplication
- The second oder induction scheme
- The second order comprehension scheme.

Note that Z_2 is far from complete even for natural statements. The scheme of projective determinacy (PD) has large cardinal strength.

Second order set theory

There is a strong analogy between models of second order arithmetic and 2-sorted structures $(V, \mathcal{C})$ with sets and classes.

These systems are analogous to arithmetic comprehension ACA_0, and Z_2:

- Gödel-Bernays class theory GBC is ZFC with global choice for 2-sorted structures.
- Kelley-Morse class theory KM is GBC with second order comprehension.

Analogies between arithmetic and set theory

	Arithmetic		Set theory
RCA_0	Computable comprehension		
WKL_0	Every infinite subtree of $2^{<\omega}$ has an infinite branch Every continuous function on $[0,1]$ attains a supremum A graph is bipartite iff it has no odd cycles		
ACA_0	Arithmetic comprehension Ramsey's theorem for triples Bolzano-Weierstrass theorem	GBC	ZFC for classes
ACA_0^+	Recursion of length ω	ETR_{Ord}	Class recursion of length Ord Any class Boolean algebra has a set-completion Any class forcing admits a forcing relation
ATR_0	Recursion along wellf. relations Clopen determinacy Any two wellordered relations are comparable	ETR	Class recursion along wellfounded class relations Clopen det. for class games ?
$\Pi^1_1 - CA_0$	Π^1_1-comprehension	$\Pi^1_1 - CA$	Π^1_1-comprehension
$\Pi^1_{<\omega} - CA_0$	Second order comprehension	KM	Second order comprehension

Differences between ATR_0 and ETR

Theorem (Steel 1977)

ATR_0 is equivalent to clopen and to open determinacy.

Theorem (Hachtmann 2016, Sato 2020)

Open class determinacy is strictly stronger than clopen class determinacy.

Theorem (Gitman, Hamkins 2017)

ETR is equivalent to clopen determinacy for class games.

Why is open (and clopen) class determinacy not provable in GBC?

- To define the non-losing strategy, one would need second order comprehension.
- The values of positions are long class wellorders.
- In the (open) truth-telling game, player I asks “is $\varphi(\vec{a})$ true” and player II answers in a way that is consistent with connectives and quantifiers. A winning strategy for II defines a truth predicate.

Question (Various proof theorists, Hamkins)

Is the statement "any two wellordered classes are comparable" equivalent to ETR?

This equivalence holds for ATR_0. It is a useful tool for reversals. The proof fails for ETR since the notion of wellorder is Π^1_1-complete in arithmetic, but Π^0_1 in set theory.

Theorem (Hamkins, Woodin 2018)

Open determinacy for class games is preserved by set forcing.

Question (Hamkins)

Is ETR preserved by set forcing?

It is not known if adding a Cohen real can destroy ETR. The problem is to show that no new class order types are added.