Krivine Realizability for Classical Set Theory

Richard Matthews

Université Paris-Est Créteil

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Joint work with Laura Fontanella and Guillaume Geoffroy

What is it all about?

- General method to produce models of ZF $(+\ \mathrm{DC})$ and even full ZFC.
- Aims to extend the Curry-Howard Correspondence from intuitionistic logic to classical logic.

Curry-Howard Correspondence

Also known as proofs-as-programs or functions-as-types.
A formal description of the relation between computer programs and mathematical proofs.

- Want classical mathematics while being able to extract some computational meaning from proofs.
- Built using a combination of Intuitionistic Realizability and Double Negation Translations.
- (Griffin) Makes use of the relation between Pierce's Law and the program *call-with-current-conditions*.

Examples

Theorem (Krivine)

It is consistent with $\mathrm{ZF} + \mathrm{DC}$ *that there exists a sequence of sets* $A_n \subseteq \mathbb{R}$ *for* $n \in \omega$ *such that*

1. *For* $n > 1$, A_n *is uncountable,*
2. *There is an injection* $f_{nm} \colon A_n \to A_m$ *iff there is a surjection* $g_{nm} \colon A_m \to A_n$ *iff* $n < m$,
3. $|A_n \times A_m| = |A_{nm}|$.

Theorem (Krivine)

It is consistent with $\mathrm{ZF} + \mathrm{DC}$ *that there exists* $X \subseteq \mathbb{R}$ *such that*

1. X *is uncountable and there is no surjection* $f \colon X \to \aleph_1$,
2. $|X| = |X \times X|$,
3. X *has a total order, every proper initial segment of which is countable,*
4. *There is a surjection* $g \colon X \times \omega_1 \to \mathbb{R}$,
5. *There is an injection* $h \colon X \times \omega_1 \to \mathbb{R}$.

Brouwer-Heyting-Kolmogorov Interpretation

- There is no proof of $\perp$.
- p is a proof of $\varphi \wedge \psi$ iff p is a pair $\langle q, r\rangle$ where q proves φ and r proves ψ.
- p is a proof of $\varphi \vee \psi$ iff p is a pair $\langle n, q\rangle$ where $n = 0$ and q proves φ or $n = 1$ and q proves ψ.
- p proves $\varphi \to \psi$ iff p is a program which transforms any proof of φ into a proof for ψ.
- p proves $\neg\varphi$ iff p proves $\varphi \to\perp$.
- p proves $\exists x\varphi(x)$ iff p is a pair $\langle a, q\rangle$ where q is a proof of $\varphi(a)$.
- p proves $\forall x\varphi(x)$ iff p is a program such that for any set a, $p(a)$ is a proof of $\varphi(a)$.

Kleene Realizability

- Developed by Kleene in 1945,
- Now seen as a realisation of the BHK interpretation,
- Gives a general method to produce *intuitionistic* models satisfying nice computer-theoretic results which are incompatible with classical logic
  - e.g. Church's thesis: If $\forall n \in \mathbb{N} \exists m \in \mathbb{N} \varphi(x, y)$ then there exists a recursive function f such that $\forall n \in \mathbb{N} \varphi(n, f(n))$.
- Let $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a primitive recursive bijection with projections $\mathbf{1^{st}}$ and $\mathbf{2^{nd}}$,
- Let $\{n\}$ be the n^{th} Turing machine (according to some fixed enumeration) and let $\{n\}(m)\downarrow$ be the assertion that the n^{th} Turing machine halts on input m.

Kleene Realizability

$$
\begin{array}{lll}
n \Vdash t = s & \text{iff} & t = s, \\
n \Vdash \varphi \wedge \psi & \text{iff} & \mathbf{1^{st}}(n) \Vdash \varphi \text{ and } \mathbf{2^{nd}}(n) \Vdash \psi, \\
n \Vdash \varphi \vee \psi & \text{iff} & \mathbf{1^{st}}(n) = 0 \text{ and } \mathbf{2^{nd}}(n) \Vdash \varphi \text{ or} \\
 & & \mathbf{1^{st}}(n) \neq 0 \text{ and } \mathbf{2^{nd}}(n) \Vdash \psi, \\
n \Vdash \varphi \to \psi & \text{iff} & \text{for every } m \in \mathbb{N}, \text{ if } m \Vdash \varphi \text{ then} \\
 & & \{n\}(m)\!\downarrow \text{ and } \{n\}(m) \Vdash \psi, \\
n \Vdash \exists x \varphi & \text{iff} & \mathbf{2^{nd}}(n) \Vdash \varphi(\mathbf{1^{st}}(n)), \\
n \Vdash \forall x \varphi(x) & \text{iff} & \text{for all } m \in \mathbb{N}, \{n\}(m)\!\downarrow \\
 & & \text{ and } \{n\}(m) \Vdash \varphi(m).
\end{array}
$$

Note

Realizers provide evidence for the "*truth*" of an assertion.

Realizability Models (McCarty)

- Start with a model of ZFC, work with the realizability structure ω.
- Define a hierarchy $\mathrm{V}(Kl)_\alpha$ by recursion on the ordinals as

$$\mathrm{V}(Kl)_\alpha := \bigcup_{\beta \in \alpha} \mathcal{P}(\omega \times \mathrm{V}(Kl)_\beta)$$

  and set $\mathrm{V}(Kl) := \bigcup_{\alpha \in \mathrm{ORD}} \mathrm{V}(Kl)_\alpha$.
- Each element, a, of the universe is a collection of pairs $\langle n, b \rangle$ where $n \in \omega$ witnesses that b is in a.
- Define $n \Vdash \varphi$ using (slight modification of) Kleene's realizability. Say that $\mathrm{V}(Kl) \Vdash \varphi$ iff $\exists n(n \Vdash \varphi)$.

Theorem (McCarty)

If $\mathrm{IZF} \vdash \varphi$ *then* $\mathrm{V}(Kl) \Vdash \varphi$.

Double Negation Translations

- Developed by Kleene in 1952 and extended to set theory by Friedman in 1973.
- Method to interpret classical mathematics in intuitionistic mathematics.
- Idea: Given φ and ψ produce two translations $\varphi^\star$ and ψ^- such that
  - If $\mathrm{IZF} \vdash \varphi$ (respectively $\mathrm{ZF} \vdash \varphi$) then $\mathrm{IZF} \setminus \mathrm{Ext.} \vdash \varphi^\star$ (respectively $\mathrm{ZF} \setminus \mathrm{Ext.} \vdash \varphi^\star$),
  - If $\mathrm{ZF} \setminus \mathrm{Ext.} \vdash \psi$ then $\mathrm{IZF} \setminus \mathrm{Ext.} \vdash \psi^-$.
- Conclusion: All four of the above theories are equiconsistent.
- The * translation works by simulating an extensional relation.
- The $^-$ translation is a $\neg\neg$-translation.

Remark

For the translation to work, $\mathrm{ZF} \setminus \mathrm{Ext.}$ should be stated with Collection (not Replacement), $\in$-Induction (not Foundation) and Weak Power Set (not Power Set).

The Theory ZF_ε

- Aim: Extract a useful theory from the $\star$ translation.
- Work in first order predicate logic without equality and only 3 binary relation symbols;
  - ε ("strong membership"),
  - $\in$ ("extensional membership"),
  - $\subseteq$ ("extensional subset").
- Define $a \simeq b$ by $(a \subseteq b) \wedge (b \subseteq a)$.

Definition (Extensionality Axioms)

$$\forall x \forall y \Big(x \in y \leftrightarrow \exists z \,\varepsilon\, y (x \simeq z) \Big)$$

$$\forall x \forall y \Big(x \subseteq y \leftrightarrow \forall z \,\varepsilon\, x (z \in y) \Big).$$

- Idea: The $\in$ relation is obtained by "collapsing" the ε operation (this is how $a \in^\star b$ is defined).

The Axioms of ZF_ε

- ε-Induction Scheme. $\forall v\big(\forall x\big(\forall y\,\varepsilon\,x\,\varphi(y,v) \to \varphi(x,v)\big) \to \forall z\varphi(z,v)\big)$.
- ε-Separation Scheme. $\forall v\forall a\exists b\forall x(x\,\varepsilon\,b \leftrightarrow (x\,\varepsilon\,a \wedge \varphi(x,v)))$.
- ε-Pairing. $\forall a\forall b\exists c(a\,\varepsilon\,c \wedge b\,\varepsilon\,c)$.
- ε-Unions. $\forall a\exists b\,\forall x\,\varepsilon\,a\,\forall y\,\varepsilon\,x(y\,\varepsilon\,b)$.
- ε-Weak Power Sets. $\forall a\exists b\forall x\,\exists y\,\varepsilon\,b\,\forall z(z\,\varepsilon\,y \leftrightarrow (z\,\varepsilon\,a \wedge z\,\varepsilon\,x))$.
- ε-Collection Scheme.
  $\forall w\forall a\exists b\,\forall x\,\varepsilon\,a\big(\exists y\varphi(x,y,w) \to \exists y\,\varepsilon\,b\,\varphi(x,y,w)\big)$.
- ε-Infinity Axiom. $\forall a\,\exists b\,\big(a\,\varepsilon\,b \wedge \forall x(x\,\varepsilon\,b \to \exists y(y\,\varepsilon\,b \wedge x\,\varepsilon\,y))\big)$.

Theorem (Friedman / Krivine)

Let $\varphi(u)$ be a formula in the language $\{\in, \simeq\}$. If $a \simeq b$ then $\mathrm{ZF}_\varepsilon \vdash \varphi(a) \to \varphi(b)$.
Suppose that $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq)$ is a model of ZF_ε. Then $(\mathrm{N}, \in, \simeq) \models \mathrm{ZF}$.

λ-Calculus

Idea: $\lambda x.t \quad \approx \quad t$ is a function on x.

Definition (λ-terms)

The class Λ is recursively defined as follows:

- (variables) $x \in \Lambda$ for any variable x,
- (application) $tu \in \Lambda$ whenever $t, u \in \Lambda$,
- (abstraction) $\lambda x.t \in \Lambda$ whenever x is a variable and $t \in \Lambda$.

Definition (β-reduction)

$(\lambda x.t)u \to_\beta t[x := u]$.

Example: (Identity) $(\lambda x.x)t \to_\beta t$.
Pseudo-example: $(\lambda x.x^2 + 1)2 \to_\beta 2^2 + 1$.

Realizability Algebras

Let κ, μ be a pair of cardinals.

Terms $\Lambda_{(\kappa,\mu)}$:

- Any variable is a term,
- (application) ts where t, u are terms,
- (*λ-abstraction*) $\lambda x.t$ where x is a variable and t a term,
- (*call-with-current-condition*) cc is a term,
- (*continuation-constant*) k_π where π is a stack,
- Special instructions t_α for $\alpha < \kappa$.

Stacks $\Pi_{(\kappa,\mu)}$:

- (*push*) $t \cdot \pi$ where t is a term and π a stack,
- Stack bottoms ω_α for $\alpha < \mu$.

Definition (Process)

A *process* is a pair $t \star \pi$ where $t \in \Lambda_{(\kappa,\mu)}$ and $\pi \in \Pi_{(\kappa,\mu)}$.

Realizers

Definition (Realizer)

A term t is called a *realizer* if it contains no occurrence of a continuation constant.[1] We denote by $\mathcal{R}$ the collection of all realizers.

Examples:

- Identity: $\mathsf{I} := \lambda x.x$,
- $\underline{0} := \lambda x.\lambda y.y$,
- $\underline{1} := \lambda x.\lambda y.xy$,
- $\underline{n} := \lambda x.\lambda y.x(x\ldots(xy))$,
- (Turing fixed point) $\lambda x.\lambda y.(y(xx))(xx)$.

[1] i.e. a k_π for some $\pi \in \Pi$

Evaluations

Definition (Evaluation)

An *evaluation* is a relation $\succ$ which satisfies the four rules

$$\begin{array}{rcll} ts \star \pi & \succ & t \star s \cdot \pi & \text{(push)}, \\ \lambda f.t \star s \cdot \pi & \succ & t[f := s] \star \pi & \text{(grab)}, \\ \mathrm{cc} \star t \cdot \pi & \succ & t \star \mathrm{k}_\pi \cdot \pi & \text{(save)}, \\ \mathrm{k}_\pi \star t \cdot \sigma & \succ & t \star \pi & \text{(restore)}. \end{array}$$

Definition (Pole)

A *pole* is a set $\perp\!\!\!\perp \subseteq \Lambda \star \Pi$ such that

$$((s \star \sigma \succ t \star \pi) \wedge (t \star \pi \in \perp\!\!\!\perp)) \rightarrow s \star \sigma \in \perp\!\!\!\perp.$$

Definition (Realizability Algebra)

A *realizability algebra* is a tuple $\mathcal{A} = (\Lambda_{(\kappa,\mu)}, \Pi_{(\kappa,\mu)}, \prec, \perp\!\!\!\perp)$.

The Realizability Structures

Recall:

$$\mathrm{V}(Kl)_\alpha := \bigcup_{\beta\in\alpha} \mathcal{P}(\omega \times \mathrm{V}(Kl)_\beta)$$

Given $\mathcal{A} = (\Lambda, \Pi, \perp\!\!\!\perp, \prec)$, define $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq)$ by

- $\mathrm{N}_0 = \emptyset$,
- $\mathrm{N}_{\alpha+1} = \mathcal{P}(\mathrm{N}_\alpha \times \Pi)$,
- $\mathrm{N}_\lambda = \bigcup_{\alpha\in\lambda} \mathrm{N}_\alpha$, for λ a limit
- $\mathrm{N} = \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{N}_\alpha$.

Idea

$(a, \pi) \in b$ provides evidence that a is not in b.
Elements of Π provide evidence for the "falsity" of a statement.

Define two sets $\|\varphi\| \subseteq \Pi$ witnessing the "falsity" of φ and $|\varphi| \subseteq \Lambda$ witnessing the "truth" of φ.

Truth and Falsity

Definition ($|\varphi|$)

$|\varphi| := \{t \in \Lambda \mid \forall \pi \in \|\varphi\| \, (t \star \pi \in \perp\!\!\!\perp)\}$.
Say t *realizes* φ ($t \Vdash \varphi$) if $t \in |\varphi|$.

Definition ($\|\varphi\|$)

- $\|\top\| = \emptyset$,
- $\|\bot\| = \Pi$,
- $\|a \not\varepsilon\, b\| = \{\pi \in \Pi \mid (a, \pi) \in b\}$,
- $\|a \notin b\| = \bigcup_{c \in \mathrm{dom}(b)} \{t \cdot t' \cdot \pi \mid (c, \pi) \in b, t \Vdash a \subseteq c, t' \Vdash c \subseteq a\}$,
- $\|a \subseteq b\| = \bigcup_{c \in \mathrm{dom}(a)} \{t \cdot \pi \mid (c, \pi) \in a,\ t \Vdash c \notin b\}$,
- $\|\varphi \to \psi\| = \{t \cdot \pi \mid t \Vdash \varphi,\ \pi \in \|\psi\|\}$,
- $\|\forall x \varphi(x)\| = \bigcup_{a \in \mathrm{N}} \|\varphi[a \setminus x]\|$.

The Model

$\mathcal{A} = (\Lambda, \Pi, \perp\!\!\!\perp, \prec)$ and $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq)$.

Definition

$\mathcal{N} \Vdash \varphi$ (φ is true in $\mathcal{N}$) if there exists a realizer $t \in \mathcal{R}$ such that $t \Vdash \varphi$.

Given a set of formulas Γ, $\mathcal{N} \Vdash \Gamma$ iff for every $\varphi \in \Gamma$, $\mathcal{N} \Vdash \varphi$.

Theorem (Krivine)

- *If* $\mathrm{ZF}_\varepsilon \vdash \varphi$ *then* $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq) \Vdash \varphi$.
- $(\mathrm{N}, \in, \simeq) \Vdash \mathrm{ZF}$.

Pierce's Law

Proposition

Suppose that $\pi \in \|\varphi\|$. Then for any ψ, $\mathrm{k}_\pi \Vdash \varphi \to \psi$.

Proof.

- Take $t \cdot \sigma \in \|\varphi \to \psi\|$.
- Then $t \Vdash \varphi$ and $\sigma \in \|\psi\|$.
- Then $\mathrm{k}_\pi \star t \cdot \sigma \succ t \star \pi$.
- But $t \star \pi \in \perp\!\!\!\perp$.
- Thus, $\mathrm{k}_\pi \star t \cdot \sigma \in \perp\!\!\!\perp$.

$\square$

Proposition (Pierce's Law)

For any φ and ψ, $\mathrm{cc} \Vdash ((\varphi \to \psi) \to \varphi) \to \varphi$.

Pierce's Law

$|\varphi| := \{t \in \Lambda \mid \forall \pi \in \|\varphi\| \, (t \star \pi \in \perp\!\!\!\perp)\}$

Proposition

Suppose that $\pi \in \|\varphi\|$. *Then for any* ψ, $\mathrm{k}_\pi \Vdash \varphi \to \psi$.

Proposition (Pierce's Law)

For any φ *and* ψ, $\mathrm{cc} \Vdash ((\varphi \to \psi) \to \varphi) \to \varphi$.

Proof.

- Suppose that $t \Vdash (\varphi \to \psi) \to \varphi$ and $\pi \in \|\varphi\|$.
- By above, $\mathrm{k}_\pi \Vdash \varphi \to \psi$.
- So $\mathrm{cc} \star t \cdot \pi \succ t \star \mathrm{k}_\pi \cdot \pi \in \perp\!\!\!\perp$. $\square$

Observation

Pierce's law is equivalent to Excluded Middle, so $\mathcal{N}$ will satisfy classical logic.

Realizability versus Forcing

- Suppose that $\mathbb{B} = (\mathbb{B}, \mathbf{1}, \mathbf{0}, \wedge, \vee, \neg)$ is a complete Boolean algebra.
- Define a realizability algebra $\mathcal{A}_{\mathbb{B}} = (\kappa, \mu, \prec, \perp\!\!\!\perp)$ as follows:
  - $\kappa = 0,\ \mu = |\mathbb{B}|$.
  - $(\omega_p \mid p \in \mathbb{B})$ is a set of stack bottoms,
  - Define a function $\tau\colon \Lambda_{(0,\mu)} \cup \Pi_{(0,\mu)} \to \mathbb{B}$ in the “obvious” way,
  - Say $t \star \pi \succ s \star \sigma$ iff $\tau(t) \wedge \tau(\pi) \leq \tau(s) \wedge \tau(\sigma)$,
  - Set $\perp\!\!\!\perp = \{t \star \pi \mid \tau(t) \wedge \tau(\pi) = 0\}$.

Theorem (M. / essentially Krivine)

For any sentence φ,

$$\mathcal{N} \Vdash_{\mathcal{A}} \varphi \quad \textit{iff} \quad \lambda x.x \Vdash_{\mathcal{A}} \varphi \quad \textit{iff} \quad \varphi \textit{ is valid in } \mathrm{V}^{\mathbb{B}}.$$

Conclusion

Every Boolean-valued model can be viewed as a realizability model.

Daleth names

Aim: Find a way to interpret every ground model set in the realizability model. In forcing, have $\check{a} := \{(\mathbb{1}, \check{b}) \mid b \in a\}$.

Definition

Given $a \in \mathrm{V}$, $\daleth(a) := \{(\daleth(x), \pi) \mid x \in a, \pi \in \Pi\}$.

Proposition

- *If* $a \subseteq b$ *then* $\mathcal{N} \Vdash \daleth(a) \subseteq \daleth(b)$,
- *If* $a \in b$ *then* $\lambda x.x \Vdash \daleth(a) \,\varepsilon\, \daleth(b)$.

Warning

$\daleth(a)$ can contain lots more elements than just $\{\daleth(b) \mid b \in a\}$.

Example

$\daleth(2)$ is a Boolean algebra of subsets of 1 in the ZF_ε model, which consistently has size greater than 2!

Ordinals

Definition (Over ZF_ε)

A set a is a ε-*ordinal* if it is a ε-transitive set of ε-transitive sets, i.e.

$$\forall x \,\varepsilon\, a \forall y \,\varepsilon\, x\, (y \,\varepsilon\, a) \quad \wedge \quad \forall z \,\varepsilon\, a \forall x \,\varepsilon\, z \forall y \,\varepsilon\, x\, (y \,\varepsilon\, z).$$

Proposition

- *If* $(\mathrm{N}, \varepsilon, \in, \subseteq) \models a$ *is a* ε*-ordinal, then* $(\mathrm{N}, \in, \simeq) \models a$ *is an ordinal.*
- *If* δ *is an ordinal in* V *then* $\mathcal{N} \Vdash$ "$\daleth(\delta)$ *is a* ε*-ordinal*".

Theorem (Fontanella, M.)

For every $n \in \omega$, $\mathcal{N} \Vdash \daleth(n)$ *is the* n^{th} *natural number.*

What about ω?

Definition

For $n \in \omega$, let $\hat{n} := \{(\hat{m}, \underline{m} \cdot \pi) \mid \pi \in \Pi, m \in n\}$.[2]

Theorem (Fontanella, M.)

For every $n \in \omega$, $\mathcal{N} \Vdash \daleth(n) \simeq \hat{n}$.

Theorem (Krivine / Fontanella, Geoffroy)

Let $\hat{\omega} = \{(\hat{n}, \underline{n} \cdot \pi) \mid \pi \in \Pi, n \in \omega\}$. *Then* $\mathcal{N} \Vdash \hat{\omega}$ *is the first infinite ordinal.*

Question

We can prove $\mathcal{N} \Vdash \hat{\omega} \subseteq \daleth(\omega)$. Does $\mathcal{N} \Vdash \hat{\omega} \simeq \daleth(\omega)$?

[2] Where $\underline{n}$ is some fixed, recursively defined sequence of realizers

Preserving Cardinals

Proposition (Over ZF_ε)

If $(\mathrm{N}, \varepsilon, \in, \subseteq) \models a$ *is a* ε*-cardinal*[3] *then* $(\mathrm{N}, \in, \simeq) \models a$ *is a cardinal.*

Theorem (Fontanella, M.)

Let $\delta > |\Lambda|$ *be a regular cardinal. Then*

$$\mathcal{N} \Vdash \forall f\, \forall a \,\varepsilon\, \daleth(\delta)\, \exists b \,\varepsilon\, \daleth(\delta) (\mathrm{Fun}(f) \to \forall y \,\varepsilon\, a (\langle y, b\rangle \,\varepsilon\, f \to \perp)).$$

i.e. for all $a \,\varepsilon\, \daleth(\delta)$ *any* $f : a \to \daleth(\delta)$ *is* *not* *a* ε*-surjection.*

Corollary

For every $\delta > |\Lambda|$, $\daleth(\delta)$ *is a* ε*-cardinal in* $\mathcal{N}$ *and hence a cardinal in* $(\mathrm{N}, \in, \simeq)$.

[3] i.e. for every $b \,\varepsilon\, a$ there is no ε-function which is an ε-surjection $(\forall y \,\varepsilon\, a\, \exists x \,\varepsilon\, b\, \langle x, y\rangle \,\varepsilon\, f)$ of b onto a.

Chain Conditions

Definition (δ-chain condition)

A realizability algebra satisfies the *δ-chain condition* if there exists a realizer $\mathbf{p} \in \mathcal{R}$ such that for every $A \subseteq \Lambda$ of cardinality at least δ, for every $t \in \Lambda$ and $\pi \in \Pi$:

if for every $a \neq b$ in A $(t \star a \cdot b \cdot \pi \in \perp\!\!\!\perp)$, then
there exists an $a \in A$ such that $\mathbf{p} \star t \cdot a \cdot \pi \in \perp\!\!\!\perp$.

Theorem (Fontanella, M.)

Suppose that a realizability algebra satisfies the δ-chain condition for some regular cardinal δ, as witnessed by the term $\mathbf{p}$. Then there exists a realizer v such that,

$$v \Vdash \forall a \,\varepsilon\, \daleth(\delta)(\text{“}\textit{there is no surjection of } a \textit{ onto } \daleth(\delta)\text{”}).$$

Chain Conditions are Chain Conditions

Definition (δ-chain condition)

A realizability algebra satisfies the *δ-chain condition* if there exists a realizer $\mathbf{p} \in \mathcal{R}$ such that for every $A \subseteq \Lambda$ of cardinality at least δ, for every $t \in \Lambda$ and $\pi \in \Pi$:

if for every $a \neq b$ in A $(t \star a \cdot b \cdot \pi \in \perp\!\!\!\perp)$, then
there exists an $a \in A$ such that $\mathbf{p} \star t \cdot a \cdot \pi \in \perp\!\!\!\perp$.

Theorem (Fontanella, M.)

Let $\mathbb{B}$ be a complete Boolean algebra and δ a regular cardinal. $\mathbb{B}$ satisfies the δ-cc if and only if $\mathcal{A}_{\mathbb{B}}$ satisfies the δ-cc.

Proof

Large Cardinals vs Large Sets

- In ZFC, large cardinals are ordinals which satisfy some nice properties.
- But in ZF_ε (or IZF), ε-ordinals are not well-behaved.[4]
- Instead easier to preserve *structural* properties derivable from large cardinals.

Theorem (ZFC)

κ is inaccessible if and only if V_κ is a model of ZF_2 (full second-order ZF).

[4] e.g. $\daleth(2)$ and $\hat{4}$ can be two distinct ordinals of size 4!

Inaccessibility

Definition (ZF_ε)

We call a set z *inaccessible* if it satisfies: ε-Empty Set, ε-Pairing, ε-Unions, ε-Infinity, ε-Weak Power Set and ε-Second-order Collection. [5]

Theorem (Fontanella, Geoffroy, M.)

Suppose that $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq)$ *satisfies* ZF_ε *plus* z *is an inaccessible set. Then* $(\mathrm{N}, \in, \simeq)$ *is a model of* ZF *plus* $z = \mathrm{V}_\delta$ *where* δ *is an inaccessible cardinal.*

Theorem (Fontanella, Geoffroy, M.)

If κ *is an inaccessible cardinal in* V *and* $\mathcal{A} \in \mathrm{V}_\kappa$, *then*

$$\mathcal{N} \Vdash \mathrm{N}_\kappa \text{ is an inaccessible set}$$

where $\mathrm{N}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{P}(\mathrm{N}_\beta \times \Pi)$.

[5] $\forall u \,\varepsilon\, z \forall f (\forall x \,\varepsilon\, u \,\exists y \,\varepsilon\, z\, (\langle x, y\rangle \,\varepsilon\, f) \to \exists v \,\varepsilon\, z \forall x \,\varepsilon\, u \,\exists y \,\varepsilon\, v\, (\langle x, y\rangle \,\varepsilon\, f))$

Measurability

- Work over ZFC and suppose κ is measurable and the critical point of the elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$.
- Suppose $\mathcal{A} \in \mathrm{V}_\kappa$.
- Define $j^\star := \{(\langle x, j(x)\rangle, \pi) \mid x \in \mathrm{N}, \pi \in \Pi\}$.

Theorem (Fontanella, Geoffroy, M.)

In $\mathcal{N} = (\mathrm{N}, \varepsilon, \in, \subseteq, j^\star)$,

- *$j^\star$ is a ε-function,*
- *$\daleth(\kappa)\,\varepsilon\, j^\star(\daleth(\kappa))$ and $\forall x\,\varepsilon\,\daleth(\kappa)(j^\star(x) \simeq x)$,*
- *$j^\star$ is an elementary embedding (for formulas in the language $\{\varepsilon, \in, \subseteq\}$),*
- *$(\mathrm{N}, \in, \simeq) \models \mathrm{ZF} +$ there exists a V-critical cardinal.*[6]

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[6] δ is a V-critical cardinal if it is the critical point of some elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$.

Reinhardt

- Suppose $(\mathrm{V}, \mathcal{C})$ is a model of GB and κ is Reinhardt (the critical point of an elementary embedding $j\colon \mathrm{V} \to \mathrm{V}$).
- Suppose $\mathcal{A} \in \mathrm{V}_\kappa$.
- Then we can define a second-order version of the realizability structure $\mathcal{N} = (\mathrm{N}, \mathcal{D}, \varepsilon, \in, \subseteq)$.

Theorem (Fontanella, Geoffroy, M.)

- $(\mathrm{N}, \mathcal{D}, \varepsilon, \in, \subseteq) \models \mathrm{GB}_\varepsilon$,
- $(\mathrm{N}, \mathcal{D}, \varepsilon, \in, \subseteq) \models j^*\colon \mathrm{N} \to \mathrm{N}$ *is an elementary embedding,*
- $(\mathrm{N}, \mathcal{D}, \in, \simeq) \models \mathrm{GB} +$ *there exists a Reinhardt cardinal.*

Open Questions

1. Does $\mathcal{N} \Vdash \hat{\omega} \simeq \daleth(\omega)$?
2. Does every realizability model satisfy SVC? [7]
3. Is there a connection between realizability models and symmetric submodels? [8]
4. Can we generalise other forcing notions to realizability (e.g. closure)?
5. Is the ground model definable in the realizability model? [9]
6. In Krivine's model for the Axiom of Choice, we know there is a realizer for Choice but what is it?
7. Can we realize the Axiom of Constructibility?
8. In the realizability model for measurable cardinals, is $j^\star$ definable in $\mathcal{N}$?

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[7] i.e. we can force Choice over the model. This would mean it is equivalent to a symmetric submodel of some model of ZFC.

[8] Conjectured to be yes by Asaf Karagila

[9] Partial positive answers by Krivine

τ Interpretation

Back

Define $\tau \colon \Lambda_{(0,\mu)} \cup \Pi_{(0,\mu)} \to \mathbb{B}$ by:

- for every stack bottom ω_p, we let $\tau(\omega_p) := p$;
- for every variable x, $\tau(x) := \tau(\mathrm{cc}) := \mathbf{1}$;
- for every term t and stack π, we let $\tau(t \cdot \pi) = \tau(t) \wedge \tau(\pi)$;
- for all λ_c-terms t, u, we let $\tau(tu) := \tau(t) \wedge \tau(u)$;
- for every variable x and every term t, we let $\tau(\lambda x \cdot t) := \tau(t)$;
- for every stack π, we let $\tau(\mathrm{k}_\pi) := \tau(\pi)$.

Observations

- If t is a realizer then $\tau(t) = \mathbf{1}$
- If $\tau(t) = \tau(s)$ then $t \Vdash_{\mathcal{A}} \varphi \Leftrightarrow s \Vdash_{\mathcal{A}} \varphi$.

$\mathbb{B}$ satisfies δ-cc implies $\mathcal{A}_{\mathbb{B}}$ satisfies δ-cc

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Proof.

- Fix $A \subseteq \Lambda$, $|A| \geq \delta$, $t \in \Lambda$, $\pi \in \Pi$. Suppose for $a \neq b$, $t \star a \cdot b \cdot \pi \in \perp\!\!\!\perp$.
- Then $0 = \tau(t) \wedge \tau(a) \wedge \tau(b) \wedge \tau(\pi)$.
- If $\tau(t) \wedge \tau(\pi) = 0$ then $\lambda f.f \star t \cdot a \cdot \pi \in \perp\!\!\!\perp$ for all $a \in A$.
- Suppose $\tau(t) \wedge \tau(\pi) > 0$. If $\exists a \in A$, $\tau(a) \wedge \tau(t) \wedge \tau(\pi) = 0$ then $\lambda f.f \star t \cdot a \cdot \pi \in \perp\!\!\!\perp$.
- Otherwise, must have $\tau(a) \wedge \tau(b) = 0$ for all $a \neq b$.
- So $\{\tau(a) \mid a \in A\}$ is an antichain of size $\geq \delta$.
- Contradicting δ-cc in $\mathbb{B}$.

$\square$

$\mathcal{A}_{\mathbb{B}}$ satisfies δ-cc implies $\mathbb{B}$ satisfies δ-cc

Back

Proof.

- Fix $A \subseteq \mathbb{B}$ to be an antichain of cardinality $\geq \delta$.
- For $p \in \mathbb{B}$, let ω_p be the stack bottom corresponding to p.
- WLOG, assume $\mathbf{0} \notin A$.
- For $a \neq b$, $\mathbf{1} \wedge a \wedge b = 0$
- So, for $a \neq b$, $\tau(\lambda f.f) \wedge \tau(\omega_a) \wedge \tau(\omega_b) \wedge \tau(\omega_1) = 0$.
- Thus $\lambda f.f \star \omega_a \cdot \omega_b \cdot \omega_1 \in \perp\!\!\!\perp$.
- So, since $|A| \geq \delta$, fix $a \in A$ such that $\mathbf{p} \star \lambda f.f \cdot \omega_a \cdot \omega_1 \in \perp\!\!\!\perp$.
- Then $0 = \tau(p) \wedge \tau(\lambda f.f) \wedge \tau(\omega_a) \wedge \tau(\omega_1) = a$, contradiction.

$\square$