

# The naïve (inconsistent) theory of truth

We add the following naïve axioms of truth to PA  
(with the schema of induction extended to  $\mathcal{L}_{\text{PA}} \cup \{T\}$ ).

$$T[a = b] \leftrightarrow a = b.$$

$$T[T[A]] \leftrightarrow T[A].$$

$$T[\neg A] \leftrightarrow \neg T[A].$$

$$T[A \wedge B] \leftrightarrow (T[A] \wedge T[B]).$$

$$T[\forall x A(x)] \leftrightarrow \forall x T[A(x)].$$

## Proposition (Tarski)

*This system is inconsistent. Because it derives:*

$$T[A] \leftrightarrow A, \text{ for all } A.$$

## Feferman's DT (the strong Kleene version)

$D[A] := T[A] \vee T[\neg A]$ . ('A has a determinate truth value')

### Axioms for $T$ :

- $\forall \vec{x} (D[\Phi(\vec{x})] \rightarrow (T[\Phi(\vec{x})] \leftrightarrow \Phi(\vec{x})))$ , for all formula  $\Phi(\vec{x})$ .
- $\forall A : D[\neg A] \rightarrow (T[\neg A] \leftrightarrow \neg T[A])$ .
- $\forall A : D[A \wedge B] \rightarrow (T[A \wedge B] \leftrightarrow (T[A] \wedge T[B]))$ .
- $\forall A : D[\forall x A(x)] \rightarrow (T[\forall x A(x)] \leftrightarrow \forall x T[A(x)])$ .

### Axioms for $D$ :

- $\forall x D[P\vec{x}]$ , for every predicate symbol  $P$  except  $T$ .
- $\forall A : D[T[A]] \leftrightarrow D[A]$ .
- $\forall A : D[A] \leftrightarrow D[\neg A]$ .
- $\forall A : D[A \wedge B] \leftrightarrow ((D[A] \wedge D[B]) \vee (D[A] \wedge \neg T[B]) \vee (D[B] \wedge \neg T[A]))$ .
- $\forall A : D[\forall x A(x)] \leftrightarrow \forall x D[A(x)]$ .

# The system CD (the strong Kleene version)

$D$ : an additional primitive predicate.

**Axioms for  $T$ :**

- $\forall \vec{x} \left( D[\Phi(\vec{x})] \rightarrow (T[\Phi(\vec{x})] \leftrightarrow \Phi(\vec{x})) \right)$ , for all formula  $\Phi(\vec{x})$ .
- $\forall A : D[\neg A] \leftrightarrow (T[\neg A] \leftrightarrow \neg T[A])$ .
- $\forall A : D[A \wedge B] \leftrightarrow (T[A \wedge B] \leftrightarrow (T[A] \wedge T[B]))$ .
- $\forall A : D[\forall x A(x)] \leftrightarrow (T[\forall x A(x)] \leftrightarrow \forall x T[A(x)])$ .

**Axioms for  $D$ :**

- $\forall x D[P\vec{x}]$ , for every predicate symbol  $P$  except  $T$  and  $D$ .
- $\forall A : D[T[A]] \leftrightarrow D[A]$ .
- $\forall A : D[A] \leftrightarrow D[\neg A]$ .
- $\forall A : D[A \wedge B] \leftrightarrow ((D[A] \wedge D[B]) \vee (D[A] \wedge \neg T[A]) \vee (D[B] \wedge \neg T[B]))$ .
- $\forall A : D[\forall x A(x)] \leftrightarrow \forall x D[A(x)]$ .
- $\forall A : D[D[A]] \leftrightarrow D[A]$ .

$$\|CD\| = \varphi_{\varepsilon_0} 0.$$

# Open problem 1

Is the following variant of CD consistent? If so how strong is it?

**Axioms for  $T$ :**

- $\forall \vec{x} \left( D[\Phi(\vec{x})] \rightarrow (T[\Phi(\vec{x})] \leftrightarrow \Phi(\vec{x})) \right)$ , for all formula  $\Phi(\vec{x})$ .
- $\forall A : D[\neg A] \leftrightarrow (T[\neg A] \leftrightarrow \neg T[A])$ .
- $\forall A : D[A \wedge B] \leftrightarrow (T[A \wedge B] \leftrightarrow (T[A] \wedge T[B]))$ .
- $\forall A : D[\forall x A(x)] \leftrightarrow (T[\forall x A(x)] \leftrightarrow \forall x T[A(x)])$ .

**Axioms for  $D$ :**

- $\forall x D[P\vec{x}]$ , for every predicate symbol  $P$  except  $T$  and  $D$ .
- $\forall A : D[T[A]] \leftrightarrow D[A]$ .
- $\forall A : D[A] \leftrightarrow D[\neg A]$ .
- $\forall A : D[A \wedge B] \leftrightarrow ((D[A] \wedge D[B]) \vee (D[A] \wedge \neg T[A]) \vee (D[B] \wedge \neg T[B]))$ .
- $\forall A : D[\forall x A(x)] \leftrightarrow \forall x D[A(x)]$ .
- $\forall A : D[D[A]]$ .

The first-order axioms of extensionality, paring, union, powerset, and infinity, and (the universal closure of) the following four second-order axioms:

$\Pi_0^1$ -CA:  $\exists X \forall x(x \in X \leftrightarrow \Phi(x))$ , for all  $\Phi \in \Pi_0^1$ ;

$\Pi_0^1$ -Sep  $\forall x \exists y(y = \{z \in x \mid \Phi(z)\})$ , for all  $\Phi \in \Pi_0^1$ ;

$\Pi_0^1$ -Ind:  $\forall x[(\forall y \in x)\Phi(y) \rightarrow \Phi(x)] \rightarrow \forall x\Phi(x)$ , for all  $\Phi \in \Pi_0^1$ ;

$\Pi_0^1$ -Repl:  $(\forall x \in y)\exists !y\Phi(x, y) \rightarrow \exists b(\forall x \in y)(\exists !y \in b)\Phi(x, y)$ , for all  $\Phi \in \Pi_0^1$

Let us neither assume Axiom of Choice (AC) nor Global Choice (GC).

# Optional axioms and MK

$\Pi_n^1$ -CA:  $\exists X \forall x(x \in X \leftrightarrow \Phi(x))$ , for all  $\Phi \in \Pi_n^1$ ;

$\Pi_n^1$ -Sep  $\forall x \exists y(y = \{z \in x \mid \Phi(z)\})$ , for all  $\Phi \in \Pi_n^1$ ;

$\Pi_n^1$ -Ind:  $\forall x[(\forall y \in x)\Phi(y) \rightarrow \Phi(x)] \rightarrow \forall x\Phi(x)$ , for all  $\Phi \in \Pi_n^1$ ;

$\Pi_n^1$ -Repl:  $(\forall x \in y)\exists !y\Phi(x, y) \rightarrow \exists b(\forall x \in y)(\exists !y \in b)\Phi(x, y)$ , for all  $\Phi \in \Pi_n^1$

## Proposition

NBG  $\vdash \Pi_n^1$ -Sep  $\rightarrow \Pi_n^1$ -Ind.

## Proposition

NBG +  $\Pi_n^1$ -CA  $\vdash \Pi_n^1$ -Sep +  $\Pi_n^1$ -Repl.

## Definition

MK := NBG +  $\bigcup_n \Pi_n^1$ -CA.  $\therefore$  MK  $\vdash \bigcup_n \Pi_n^1$ -Sep +  $\bigcup_n \Pi_n^1$ -Repl.

## Open problem 2

### Theorem (Mostowski?)

$\text{MK} + \text{AC}$  is equiconsistent with  $\text{MK}$ .

### Theorem

$\text{NBG} + \Pi_n^1\text{-CA} + \text{AC}$  is equiconsistent with  $\text{NBG} + \Pi_n^1\text{-CA}$ .

### Question 2

Is  $\text{NBG} + \Pi_n^1\text{-CA} + \bigcup_n \Pi_n^1\text{-Sep} + \bigcup_n \Pi_n^1\text{-Repl} + \text{AC}$  equiconsistent with the same theory without AC?

### Facts

- $\text{NBG} + \bigcup_n \Pi_n^1\text{-Sep} + \bigcup_n \Pi_n^1\text{-Repl} + \text{AC}$  is equiconsistent with the same theory without AC.
- Adding  $\Sigma_1^1\text{-Coll}$ , FP, LFP, etc., are still equiconsistent with the original theory without AC. (?)