

# Algorithmically Random Trigonometric Series

Rod Downey  
Victoria University  
Wellington  
New Zealand

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# Uses of Algorithmic Randomness in Mathematics

- ▶ Quite aside from its intrinsic interest, there have recently been a number uses of **algorithmic** randomness in mathematics.
- ▶ After all, giving meaning to randomness for an individual string or sequence (real) is surely a useful tool.
- ▶ Lutz, Stull and Lutz have had great success using this tool and their “point to set” principle to prove theorems in geometric measure theory.
- ▶ New simpler proofs of old results: Lutz and Lutz gave a new proof of of the two-dimensional case of the well-known Kakeya conjecture.
- ▶ In work on subshifts of finite type, Hochman and Meyerovitch, showed that values of entropies of subshifts of finite type over  $\mathbb{Z}^d$  for  $d \geq 2$  are exactly the complements of halting probabilities.
- ▶ Often algorithms result from using probability to show that something exists and then observing that the condition is a closed one. E.g. Bosserhof’s construction in Banach Spaces.
- ▶ In this paper we will show that understanding **how much randomness** is needed for classical theorems can yield algorithms.

# Motivating Questions

- ▶ Bollobás in the introduction to his book, originally in 1985. In this introduction, Bollobás motivates the use of probabilistic ideas in graph theory. He mentioned that earlier probabilistic application had been found in analysis via three famous papers of Paley and Zygmund (1932-33)

*"Paley and Zygmund (1930a,b,1932) had investigated random series of functions. One of their results was that if the real numbers  $c_n$  satisfy  $\sum_{n=0}^{\infty} c_n^2 = \infty$  then  $\sum_{n=0}^{\infty} \pm c_n \cos nx$  fails to be a Fourier-Lebesgue series for almost all choices of the signs. To exhibit a sequence of signs with this property is surprisingly difficult: indeed there is no algorithm known which constructs an appropriate sequence of signs from any sequence  $c_n$  with  $\sum_{n=0}^{\infty} c_n^2 = \infty$ ."*

- ▶ An almost identical question can be found even earlier in the 1968 version of Kahane's book (most recently, (2003 version) page 47),

*"If  $\sum c_n^2 = \infty$ , there exists a choice of signs  $\pm$  such that  $\sum \pm c_n \cos(nt + \varphi_n)$  is not a Fourier-Stieltjes series. A surprising fact is that nobody knows how to construct these signs explicitly, but a random choice works."*

- ▶ Thus, this natural question is now at least 50 years old.

# Formulating this mathematically

- ▶ As logicians we know how to formulate questions like this.
- ▶ We wish to say that **given** the data specifying the input we have an **algorithm** to specify the signs.
- ▶ We can use the umbrella of computable analysis to work within.
- ▶ A positive solution to Bollobás's problem would consist of an algorithm which runs on Turing's idealised machine.
- ▶ On an “input tape” of the machine is written the sequence of reals  $\langle c_n \rangle$ . The machine runs indefinitely, and on an “output tape” is gradually written a solution: a sequence  $\langle x_n \rangle \in \{-1, 1\}^\infty$  such that  $\sum x_n c_n \cos nt$  is not a Fourier-Lebesgue series.
- ▶ The main point is that there is a single algorithm which given the input  $\langle c_n \rangle$  produces a desired output  $\langle x_n \rangle$ . We say that the outputs are *uniformly computable* from the inputs.

# Strategy

- ▶ Identify the amount of randomness needed for the proof of the classical theorem, and use that to understand the existence or nonexistence of an operator.
- ▶ We use the theory of **algorithmic randomness**.
- ▶ This theory abandons the idea of absolute randomness and thinks of a real as being random at a certain level of sensitivity if it passes all tests for that level.

## Definition

- (i) A *name of an open set*  $U$  is a list  $\langle V_0, V_1, V_2, \dots \rangle$  of basic open sets such that  $U = \bigcup_n V_n$ .<sup>a</sup>
- (ii) A *name of a sequence of open sets*  $U_0, U_1, \dots$  is a sequence consisting of a name of  $U_0$ , a name of  $U_1$ ,  $\dots$ .
- (iii) A *name of a  $G_\delta$  set*  $G$  is a name of a nested sequence of open sets  $U_0, U_1, U_2, \dots$  such that  $G = \bigcap_n U_n$ .
- (iv) A name of an  $F_\sigma$  set is a name of its complement.

- ▶ We analysed a theorem of Potgeiter

## Theorem

*Given  $\langle c_n \rangle$  and  $\langle \varphi_n \rangle$  with  $\sum c_n^2 = \infty$ , we can compute a name of a null  $F_\sigma$  set containing all  $x \in \{-1, 1\}^\infty$  for which  $\sum x_n c_n \cos(nt + \varphi_n)$  is a Fourier-Stieltjes series.*

- ▶ We can then quote a standard result from computability theory:

## Theorem

*Given a (computable) name of a null  $F_\sigma$  set  $H$ , we can compute a (computable) point  $x \notin H$ .*

- ▶ That's because they are **Kurtz Tests**. Recall a Kurtz null test (e.g. in  $2^\omega$  is a computable collection of clopen sets  $\mathcal{V} = \{V_e \mid e \in \mathbb{N}\}$  with  $V_e = \{[\sigma] : \sigma \in D_{f(e)}\}$ , with  $f$  computable and  $\lambda(V_e) \leq 2^{-e}$ .
- ▶ Running the enumeration of  $\cap \mathcal{V}$  till the measure is below  $2^{-(e+1)}$  allows us to compute a Cauchy sequence converging to a computable real, outside of  $\cap \mathcal{V}$  to within  $2^{-e}$ .
- ▶ The same proof works for **Schnorr Tests** which are ML-tests, but with



# Rademacher Series

- ▶ The Paley-Zygmund theorems were motivated by questions of Rademacher, who, along with Steinhaus, seem to be the original people to study random series.
- ▶ Random trigonometric series arise quite naturally in, for example, Brownian motion, and random noise in image processing
- ▶ Major area of analysis.

## Theorem (Rademacher 1922)

*Let  $\{c_n\}$  be a sequence of real numbers.*

- (i) If  $\sum c_n^2 = \infty$  then  $\sum x_n c_n$  diverges for almost all  $x \in \{-1, 1\}^\infty$ .*
- (ii) If  $\sum c_n^2 < \infty$  then  $\sum x_n c_n$  converges for almost all  $x \in \{-1, 1\}^\infty$ .*

- ▶ The canonical nontrivial example for (ii) is the Harmonic series  $\sum_n \frac{1}{n}$ .
- ▶ If  $\sum c_n^2 = \infty$ , then choosing  $x_n$  so as to make  $x_n c_n > 0$  will cause divergence of the Rademacher series.
- ▶ But what level of randomness is needed?

## Theorem

Let  $\langle c_n \rangle$  be a sequence of real numbers and let  $x = \langle x_n \rangle \in \{-1, 1\}^\infty$ .

- (i) If  $\sum c_n^2 = \infty$  and  $x$  is Kurtz random relative to  $\langle c_n \rangle$  then  $\sum x_n c_n$  diverges.
- (ii) If  $\sum c_n^2 < \infty$  and  $x$  is Schnorr random relative to  $(\langle c_n \rangle, \sum c_n^2)$  then  $\sum x_n c_n$  converges.

- ▶ Part (ii) was first shown by Ongay-Valverde and Tveite (2021). Potgieter (2018) showed that ML-randomness suffices for both cases. We will give simple proofs of both.
- ▶ We also consider what happens if we don't know the value of  $\sum c_n^2$ .

# Divergence of Rademacher Series

- ▶ Given  $\langle c_n \rangle$  for which  $\sum c_n^2 = \infty$  we can (uniformly) compute a name of a null  $F_\sigma$  set containing all  $x \in \{-1, 1\}^\infty$  for which  $\sum x_n c_n$  converges.
- ▶ **Paley-Zigmund Inequality** For any natural  $N$  and sequence of reals  $a_0, a_1, \dots, a_{N-1}$ , if  $\sum_{n < N} a_n^2 > 1/4$  then

$$\mathbb{P}\left\{\tau \in \{-1, 1\}^N : \left| \sum_{n < N} \tau_n a_n \right| > \frac{1}{2}\right\} > \frac{1}{6}, \quad (1)$$

where  $\mathbb{P}$  denotes the fair-coin probability measure on  $\{-1, 1\}^N$ .

- ▶ Given  $\langle c_n \rangle$  with  $\sum c_n^2 = \infty$  we can compute a partition of  $\mathbb{N}$  into intervals  $I_0 < I_1 < \dots$  (so  $\min I_{k+1} = \max I_k + 1$ ), with each interval  $I_i$  sufficiently long so that

$$\sum_{n \in I_i} c_n^2 > \frac{1}{4}.$$

- For each  $i$  let

$$C_i = \left\{ x \in \{-1, 1\}^\infty : (\forall j \geq i) \left| \sum_{n \in I_j} x_n c_n \right| \leq \frac{1}{2} \right\}.$$

- Then each  $C_i$  is closed and null (it is the product of infinitely many independent clopen sets, each with measure at most  $1/6$ ). Hence,  $H = \bigcup_i C_i$  is a null  $F_\sigma$  set with  $\langle c_n \rangle$ -computable name, that contains every  $x$  for which  $\sum x_n c_n$  converges.

# Convergence of Rademacher Series

- **Paley-Zigmund** For any  $N$ , sequence of real numbers  $\langle a_n \rangle_{n < N}$  and any  $\epsilon > 0$ ,

$$\mathbb{P}\left\{\tau \in \{-1, 1\}^N : \max_{m < N} \left| \sum_{n \leq m} \tau_n a_n \right| > \epsilon\right\} \leq \frac{1}{\epsilon^2} \sum_{n < N} a_n^2. \quad (2)$$

- The inequality holds for  $N = \infty$  as well, in which case we need of course to replace max with sup. With the triangle inequality, we can deduce the following:

$$\mathbb{P}\left\{\tau \in \{-1, 1\}^N : \max_{k \leq m < N} \left| \sum_{n=k}^m \tau_n a_n \right| > \epsilon\right\} \leq \frac{4}{\epsilon^2} \sum_{n < N} a_n^2. \quad (3)$$

(In fact, the proof of Kolmogorov's inequality gives the bound  $\sum a_n^2 / \epsilon^2$ .)

- ▶ Given both a name of a nested sequence  $\langle U_n \rangle$  of open sets such that  $\lambda(U_n) \rightarrow 0$ , and the sequence  $\langle \lambda(U_n) \rangle$ , we can compute a Schnorr name of  $\bigcap_n U_n$ .
- ▶ The proof is an easy induction.
- ▶ Given  $\langle c_n \rangle$  for which  $\sum c_n^2 < \infty$ , and the value of that sum, we can (uniformly) compute a Schnorr name of a null set containing all  $x \in \{-1, 1\}^\infty$  for which  $\sum x_n c_n$  diverges.
- ▶ Given  $\langle c_n \rangle$  and  $\sum c_n^2$ , we can compute a partition of  $\mathbb{N}$  into intervals  $I_0 < I_1 < \dots$  such that for all  $k \geq 1$ ,  $\sum_{n \in I_k} c_n^2 < 2^{-3k-2}$ ,
- ▶ y (3),  $\lambda(A_k) \leq 2^{-k}$ , where

$$A_k = \left\{ x \in \{-1, 1\}^\infty : \max_{J \subseteq I_k} \left| \sum_{n \in J} x_n c_n \right| > 2^{-k} \right\},$$

- ▶ Let  $U_m = \bigcup_{k > m} A_k$ . A name of  $\langle U_m \rangle$  can be obtained computably given the data, and  $\lambda(U_m)$  is computable as well given the data ( $U_{m,s} = \bigcup_{k=m+1}^s A_k$  is a clopen set approximating  $U_m$  to within  $2^{-s}$ ). If  $x \in \{-1, 1\}^\infty$  and  $\sum x_n c_n$  diverges then  $x \in A_k$  for infinitely many  $k$ , so  $x \in \bigcap_m U_m$ .

- ▶ The method of Potgieter only gives ML tests and it is not clear if his proof can be adapted to give Schnorr tests. (Detailed analysis in our full paper.)
- ▶ Ongay-Valverde and Tveite (2021), Lemma 6.7 claim to prove (ii). They use sophisticated machinery developed by Rute in an unpublished manuscript, rather than directly producing Schnorr null sets. However, it appears that they only prove convergence of a subsequence of the partial sums  $\sum_{n \leq k} x_n c_n$ .
- ▶ **Question:** What if we are given a sequence  $\langle c_n \rangle$  with  $\sum c_n^2 < \infty$ , but we are not told what the sum is?
- ▶ It appears that Schnorr randomness will not suffice in this case.

## Definition

An *OW-null set* is a set contained in an intersection  $\bigcap_n U_n$ , where  $\langle U_n \rangle$  is a nested sequence of uniformly enumerable open sets such that for some left-c.e. real  $\alpha$  and some increasing computable rational approximation  $\langle \alpha_n \rangle$  of  $\alpha$ , we have  $\lambda(U_n) \leq \alpha - \alpha_n$  for all  $n$ .

- ▶  $\lambda(U_n) \rightarrow 0$  is witnessed by the fact that the approximation  $\alpha_n \rightarrow \alpha$  converges. Computably, at very late stages  $s$ , we discover that the sets  $U_n$  for  $n < s$  are “allowed to grow” by a large amount (much larger than  $2^{-s}$ ).
- ▶ This “amount of growing” eventually goes to 0, but we cannot tell computably how quickly.



- We could prove the following

## Theorem

*Let  $\langle c_n \rangle$  be such that  $\sum c_n^2 < \infty$ . If  $x \in \{-1, 1\}^\infty$  is OW-random relative to  $\langle c_n \rangle$ , then  $\sum x_n c_n$  converges.*

- The proof is to observe

$$U_m^\epsilon = \left\{ x \in \{-1, 1\}^\infty : \sup_{k \geq m} \left| \sum_{n=m}^k x_n c_n \right| > \epsilon \right\}.$$

is an OW-test.

# Fourier-Stieltjes Series

- ▶ We are given  $\langle c_n \rangle$  and  $\langle \varphi_n \rangle$ . For each finite binary string  $\tau = (\tau_0, \tau_1, \dots, \tau_m) \in \{-1, 1\}^{m+1}$ , let the corresponding *Fejér sum* be

$$\sigma_\tau(t) = \sum_{n \leq m} \left(1 - \frac{n}{m}\right) \tau_n c_n \cos(nt + \varphi_n).$$

- ▶ This is a continuous function on  $[0, 2\pi]$  and the functions  $\sigma_\tau$  for  $\tau \in \{-1, 1\}^{<\infty}$  are *uniformly computable* relative to  $(\langle c_n \rangle, \langle \varphi_n \rangle)$ .
- ▶ By Zygmund 1959, for all  $x \in \{-1, 1\}^\infty$ ,  $\sum x_n c_n \cos(nt + \varphi_n)$  is Fourier-Stieltjes if and only if

$$\sup_m \|\sigma_{x \upharpoonright m}\|_1 < \infty,$$

where recall that  $\|f\|_1 = \int_0^{2\pi} |f(t)| dt$ .

- By Pour-El and Richards, Ch 0, Thm 5, the values  $\|\sigma_\tau\|_1$  are uniformly computable relative to the data. For each  $K$ , let

$$C_K = \{x \in \{-1, 1\}^\infty : (\forall m) \|\sigma_{x \upharpoonright m}\|_1 \leq K\}.$$

Then each  $C_K$  is closed, effectively so given the data. The required  $F_\sigma$  set is thus  $\bigcup_K C_K$ ; this set is null by the classical result that under the assumption,  $\sum x_n c_n \cos(nt + \varphi_n)$  is not Fourier-Stieltjes for almost all  $x$ .

- We remark that Potgeiter (2018) follows similar path, but the proof has a gap in concerning Reimann sums and we repaired this with the Pour-El Richards material.
- **Question:** What about pointwise convergence/divergence?

# Pointwise results

## Theorem (Paley and Zygmund 1932)

*Let  $\langle c_n \rangle$  and  $\langle \varphi_n \rangle$  be a sequences of real numbers.*

- (i) If  $\sum c_n^2 < \infty$ , then for almost all  $x \in \{-1, 1\}^\infty$ ,  $\sum x_n c_n \cos(nt + \varphi_n)$  converges for almost all  $t \in [0, 2\pi]$ .*
- (ii) If  $\sum c_n^2 = \infty$ , then for almost all  $x \in \{-1, 1\}^\infty$ ,  $\sum x_n c_n \cos(nt + \varphi_n)$  diverges for almost all  $t \in [0, 2\pi]$ .*

- ▶ We study effectiveness, asking not only for almost everywhere divergence, but also, what level of randomness of  $t$  ensures this divergence. This leads us to consider randomness in the product space  $\{-1, 1\}^\infty \times [0, 2\pi]$ , which is defined as expected, using the product measure  $\lambda \times \mu$ .

## Theorem

Let  $\langle c_n \rangle$  and  $\langle \varphi_n \rangle$  be sequences of real numbers, and suppose that  $\sum c_n^2 = \infty$ . If  $(x, t) \in \{-1, 1\}^\infty \times [0, 2\pi]$  is Schnorr random relative to  $(\langle c_n \rangle, \langle \varphi_n \rangle)$  then  $\sum x_n c_n \cos(nt + \varphi_n)$  diverges.

We note that this theorem implies that if  $x$  is Schnorr random then  $\sum x_n c_n \cos(nt + \varphi_n)$  diverges almost everywhere.

## Theorem

Let  $\langle c_n \rangle$  and  $\langle \varphi_n \rangle$  be sequences of real numbers, and suppose that  $\sum c_n^2 < \infty$ . If  $(x, t) \in \{-1, 1\}^\infty \times [0, 2\pi]$  is Schnorr random relative to  $(\langle c_n \rangle, \langle \varphi_n \rangle, \sum c_n^2)$  then  $\sum x_n c_n \cos(nt + \varphi_n)$  converges.

- ▶ The proofs are along similar lines of analysing the random sets obtained in variations of the classical proofs.
- ▶ We don't know whether Kurtz randomness suffices for divergence.
- ▶ There are many open lower bound questions:

# Lower Bounds

- We'd like to have classifications like:

## Theorem (Bratkka, Miller, Nies (2016))

*A point  $x \in [0, 1]$  is ML-random if and only if every computable function  $f: [0, 1] \rightarrow \mathbb{R}$  of bounded variation is differentiable at  $x$ .*

## Theorem (Gács, Hoyrup, Rojas (2011))

*Let  $(X, \mu)$  be a computable measure space, and let  $T: X \rightarrow X$  be computable and ergodic. A point  $x \in X$  is Schnorr random if and only if for every computable function  $f: X \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i < n} f(T^i x) = \int f d\mu.$$

- For instance, does Kurtz randomness characterize divergence of Rademacher series?

- ▶ NO
- ▶ This is a **new (and natural) phenomenon** in randomness:

## Theorem

*Suppose that  $P \subset \{-1, 1\}^\infty$  is effectively closed, and that there is a computable tree  $T \subset \{-1, 1\}^{<\infty}$  such that  $P = [T]$  and for all  $n$ ,  $T$  contains fewer than  $\log_2 n$  many strings of length  $n$ . Then there is a computable sequence  $\langle c_n \rangle$  such that  $\sum c_n^2 = \infty$ , but  $\sum x_n c_n$  converges for all  $x \in P$ .*

- ▶ Such an effectively closed set must be null, as  $\log_2 n / 2^n \rightarrow 0$ , and so (as is necessary) no  $x \in P$  is Kurtz random. We note that very small effectively closed sets of Binns (2005) have this property.
- ▶ The proof: Let  $I_k = [2^k, 2^{k+1})$ . For each  $k$ , since there are at most  $k$  strings of length  $2^{k+1}$  in  $T$ , there is some  $n_k \in I_k$  such that  $\tau_{n_k}$  is a constant value  $i_k$  for all  $\tau \in T$  of length  $2^{k+1}$ . We let  $c_{n_k} = (-1)^k i_k / \sqrt{k}$  and  $c_n = 0$  if  $n \neq n_k$  for all  $k$ .



- ▶ The following lower bound is also weaker than randomness. A sequence  $x \in \{-1, 1\}^\infty$  is **bi-immune** if neither  $\{n : x_n = 1\}$  nor its complement  $\{n : x_n = -1\}$  contain an infinite computable set (equivalently, an infinite computably enumerable set). All Kurtz random sequences are bi-immune.

### Theorem

*If  $x$  is not bi-immune then there is a computable sequence  $\langle c_n \rangle$  with  $\sum c_n^2 = \infty$  but  $\sum x_n c_n$  converges.*

- ▶ The proof: Let  $A$  be an infinite computable set such that either  $x_n = 1$  for all  $n \in A$ , or  $x_n = -1$  for all  $n \in A$ . Let  $n_1, n_2, \dots$  be the increasing enumeration of the elements of  $A$ . Let  $c_{n_k} = (-1)^k / \sqrt{k}$ ; if  $n \neq n_k$  for any  $k$  let  $c_n = 0$ .

## Some Recent Results

- ▶ Define SCP **strong convergence property**:  $\sum x_i a_i$  converge whenever  $\langle a_i \rangle$  computable and square-summable, even if the sum of squares is not computable.
- ▶ (Downey, Greenberg, Tanggarra) Schnorr Random implies CP (where the sum is known)
- ▶ (Bienvenu and Greenberg) Computably Random implies SCP
- ▶ Motivated by several counterexamples, Ruofei Xie looked at strong extensions: expanding the area for selecting the sequence of reals from the computable ones to the partial computable ones. The formal definition is as follows. Definition 5.1. Given a partial computable function  $f$ , let  $\bar{f}$  be the extension which converges and gives 0 of  $f(x) \uparrow$ . Now we seek convergence if  $\sum x_n \bar{f}(n)$  converges for all partial  $f$  with  $\sum_n \bar{f}(n)^2 < \infty$ . Call this VSCP.
- ▶ (Ruofei Xi) MLR implies VSCP
- ▶ All notions distinct.

# References

- ▶ *Algorithmically random series, and uses of algorithmic randomness in mathematics*, Downey, Greenberg and Tanggarra, submitted.
- ▶ Béla Bollobás. *Random graphs*, Cambridge University Press, Cambridge, second edition, 2001.
- ▶ Vasco Brattka, Joseph S. Miller, and André Nies. Randomness and differentiability. *Trans. Amer. Math. Soc.*, 368(1):581–605, 2016.
- ▶ Peter Gács, Mathieu Hoyrup, and Cristóbal Rojas. Randomness on computable probability spaces—a dynamical point of view. *Theory Comput. Syst.*, 48(3):465–485, 2011.
- ▶ Iván Ongay-Valverde and Paul Tveite. Computable analogs of cardinal characteristics: prediction and rearrangement. *Ann. Pure Appl. Logic*, 172(1):Paper No. 102872, 23, 2021.
- ▶ Jean-Pierre Kahane. *Some random series of functions*. Cambridge University Press, 2003.
- ▶ Marian B. Pour-El and J. Ian Richards. *Computability in Analysis and Physics*. Cambridge University Press, 2017.

- ▶ Paul Potgieter. Algorithmically random series and brownian motion. *Annals of Pure and Applied Logic*, 169(11):1210–1226, 2018.
- ▶ R. E. A. C. Paley and A. Zygmund. On some series of functions (1)-(3). *Mathematical Proceedings of the Cambridge Philosophical Society*, 26(4):337–257, 1930-32.
- ▶ H. Rademacher. Einige sätze über reihen von allgemeinen orthogonalfunktionen. *Mathematische Annalen*, 87:112–138, 1922.
- ▶ Hugo Steinhaus. Über die wahrscheinlichkeit dafür, das der konvergenzkreis einer potenzreihe ihre naturliche grenze ist. *Mathematische Zeitschrift*, 31(1):408–416, 1930.

Thank You