Conjugacy, classification, and complexity

Two-Day Logic Meeting
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Structures and automorphism groups

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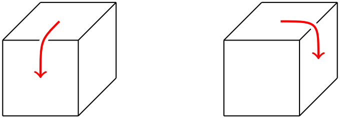
Definition

We will say two automorphisms $\phi, \psi \in \mathrm{Aut}(M)$ are the same kind if they are conjugate: there exists $\alpha \in \mathrm{Aut}(M)$ such that $\psi = \alpha\phi\alpha^{-1}$.

A finite example

Example

Let $M =$ the cube graph. Any two quarter turns are conjugate:



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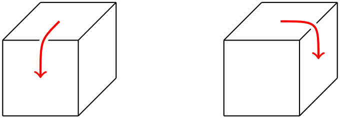

There are ten conjugacy classes: identity (1x), quarter turns about a face (6x), half turns about a face (3x), one-third turns about a vertex (8x), half turns about an edge (6x), and... each of these composed with inversion $x \mapsto -x$.

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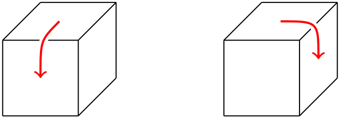

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Group theorists write the class equation:
$|\operatorname{Aut}(M)| = 1 + 6 + 3 + 8 + 6 + 1 + 6 + 3 + 8 + 6.$

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$$(01)(234)(5678)\cdots \qquad\qquad (02)(135)(2468)\cdots$$

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We can therefore classify automorphisms ϕ of K_∞ by the complete, concrete, and explicitly calculated invariants $t_\phi =$ the sequence of numbers recording the cycle type of ϕ.

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To answer this, we examine several examples using the lens of Borel complexity theory of equivalence relations.

Given an equivalence relation like conjugacy, we can measure its complexity by locating it in the Borel reducibility hierarchy:

Definition

An equivalence relation E on X is Borel reducible to F on Y, written $E \leq_B F$, if there is a Borel function $f\colon X \to Y$ such that

$$x \, E \, x' \iff f(x) \, F \, f(x')$$

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\mathrm{Id} \arrow[r, dash] & E_0 \arrow[r, dash, bend left] \arrow[r, dash, bend right] & E_\infty \arrow[r, dash] & \mathrm{Id}^+ \arrow[r, dash] & E_{\mathrm{complete}}
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- Here, Id is the equality relation on $\mathbb{N}^{\mathbb{N}}$ and corresponds to the case of concrete invariants.

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The conjugacy relation on $\mathrm{Aut}(K_\infty^+)$ *is Borel bireducible with* Id^+*, the set equality equivalence relation on sequences of elements of* $\mathbb{N}^\mathbb{N}$*.*

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Remark

Hence, we cannot classify automorphisms of K_∞^+ by complete, concrete, explicitly calculated invariants.

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Example

Let $M = G_\infty$ be the random graph, that is, the countable graph which is universal (contains every countable graph) and homogeneous (finite partial automorphisms extend to automorphisms).

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Remark

We have shown many random structures have an automorphism classification which is complete. The first example was $\mathrm{Aut}(\mathbb{Q}, <)$, due to Foreman.

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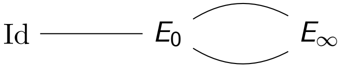
Remark

The structure of the countable Borel equivalence relations is simple at the bottom and top, and wild in the middle:



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Definition

Let Γ be a countable group. Then E_Γ denotes the Bernoulli equivalence relation on 2^Γ defined by $x \sim y$ if there exists $\gamma \in \Gamma$ such that:

$$(\forall \alpha \in \Gamma)\ x(\gamma\alpha) = y(\alpha)$$

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Theorem (Thomas)

There exists a family $\mathcal{F}$ continuum many groups such that E_Γ for $\Gamma \in \mathcal{F}$ are pairwise Borel incomparable, and hence intermediate between E_0 and E_∞.

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- By a construction of Olshanskii (plus...), there exists a family $\mathcal{F}$ of continuum many pairwise nonisomorphic countable, simple, quasifinite, property (T) groups.
- By a result of Popa, there is "superrigidity" for Bernoulli actions of property (T) groups. This means that a Borel homomorphism from E_Γ to $E_{\Gamma'}$ (really, its free part) gives rise to a group homomorphism $\Gamma \to \Gamma'$.

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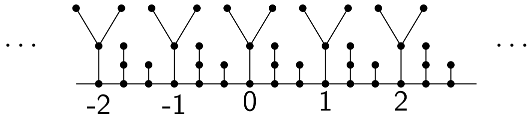
Example (The graph $G_{\mathbb{Z}}$)



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Theorem

There exists a family $\mathcal{F}$ of continuum many pairwise Borel incomparable conjugacy relations intermediate between E_0 and E_∞.

Thank you!