# Positivity of L-functions and “Completion of square"

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# Outline

# Riemann Hypothesis (RH)

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

$$= \prod_{p,\ \text{primes}} \frac{1}{1 - p^{-s}}, \quad s \in \mathbb{C}, \ \mathrm{Re}(s) > 1$$

Analytic continuation and Functional equation

$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s), \quad s \in \mathbb{C}$$

**Conjecture**

*The non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the line*

$$\mathrm{Re}(s) = \frac{1}{2}.$$

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# Equivalent statements of RH

- Let
$$\pi(X) = \#\{\text{primes numbers } p \leq X\}.$$
Then
$$RH \iff \left|\pi(X) - \int_2^X \frac{dt}{\log t}\right| = O(X^{1/2+\epsilon}).$$

- Let
$$\theta(X) = \sum_{p<X} \log p.$$
Then
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# RH for a curve $C$ over a finite field $\mathbb{F}_q$

**Theorem (Weil)**

$$|\#C(\mathbb{F}_{q^n}) - (1 + q^n)| \leq 2\, g_C \sqrt{q^n}.$$

**Remark**

To compare with the case for $\mathbb{Q}$:

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# More L-functions

## Example

To an elliptic curve over $\mathbb{Q}$

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Hasse–Weil associates an L-function

$$L(s, E) = \prod_{p\text{: good}} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

where, for good $p$

$$a_p = p + 1 - E(\mathbb{F}_p).$$

# Grand Riemann Hypothesis (GRH)

There are many other L-functions, e.g., those attached automorphic representations on $\mathrm{GL}(N)$.

**Conjecture**

*Nontrivial zeros of all automorphic L-functions lie on the line*

$$\mathrm{Re}(s) = \frac{1}{2}.$$

# A corollary to Riemann Hypothesis

Suppose that an $L$-function has the following properties

- $L(s)$ is an entire function.
- $L(s) \in \mathbb{R}$ if $s$ is real.
- $L(s) > 0$ as $s \in \mathbb{R}$ and $s \to \infty$.

We have

$$GRH \Longrightarrow L(1/2) \geq 0,$$

or more generally, the first non-zero coefficient (i.e., the leading term) in the Taylor expansion is positive.

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# Superpositivity: non-leading terms

**Lemma (Stark–Zagier (1980), Yun–Zhang)**

*Let $\pi$ be a self-dual cuspidal automorphic representation of $\mathrm{GL}_n$. Normalize its functional equation such that*

$$L(s,\pi) = \pm L(1-s,\pi).$$

*Then*

$$GRH \Longrightarrow L^{(r)}(1/2,\pi) \geq 0, \text{ for all } r \geq 0.$$

Here

$$L(s,\pi) = \sum_{r=0}^{\infty} L^{(r)}(1/2,\pi)\,\frac{(s-1/2)^r}{r!}.$$

## The idea of proof

Hadamard product expansion (and the functional equation and the self-duality)

$$L(s+1/2) = c \cdot s^r \prod_{\rho} \left(1 - \frac{s^2}{\rho^2}\right),$$

- the product runs over all the zeros $\frac{1}{2} \pm \rho$ of $L(s)$ such that $\rho \neq 0$,
- $r = \mathrm{ord}_{s=1/2} L(s)$, and $c > 0$ is the leading Taylor coefficient.

Now note that

$$GRH \iff \mathrm{Re}(\rho) = 0.$$

# Super-postivity of L-functions

- Super-postivity does not imply GRH. But it implies the non-existence of Landau–Siegel zero.
- Known for Riemann zeta function (Polya, 1927). Sarnak introduced a notion of “positive definite" for L-functions. If an L-function is positive definite then it is “super-positive". Not known if there are infinitely many positive definite L-functions.
- Goldfeld–Huang: there are infinitely many “super-positive" automorphic L-functions for $\mathrm{GL}(2)$.

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# Completion of square

The super-positivity suggests us

to express $L^{(r)}(1/2, \pi)$ in terms of some “squared quantity".

We explain two such examples

- (Gross–Zagier, Yuan–Zhang–Zhang) The first derivative

$$L'(1/2, \pi) \geq 0$$

  if $\pi$ appears in the cohomology of Shimura curve over a (totally real) number field $F$.
- (Yun–Zhang) “Higher Gross–Zagier formula" over function fields.

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- (Yun–Zhang) “Higher Gross–Zagier formula" over function fields.

# Gross–Zagier formula

**Theorem**

*Let $E$ be an elliptic curve over $\mathbb{Q}$. There is a point $P \in E(\mathbb{Q})$ such that*

$$L'(1, E) = c \cdot \langle P, P \rangle,$$

*where the RHS is the Néron–Tate height pairing*

$$\langle \cdot, \cdot \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$

*and $c$ is a positive number.*

The point $P$ in the above formula is the so-called Heegner point. The Néron–Tate height pairing is known to be positive definite. Hence

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# Heegner points on modular curve

- The modular curve $X_0(N)$ is moduli space classifying elliptic curves with auxiliary structure:

$$\begin{array}{ccc} X_0(N) & \longrightarrow & E \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} \mathbb{Q} & \end{array}$$

- The Heegner points are represented by those special elliptic curves with complex multiplication.

# Drinfeld Shtukas

Now fix $k = \mathbb{F}_q$, and $X/k$ a smooth geometrically connected curve. We consider the moduli stack of Drinfeld Shtukas of rank $n$. For a $k$-scheme $S$, we have

$$\mathrm{Sht}^r_{\mathrm{GL}_n,X}(S) = \left\{ \begin{array}{c} \text{vector bundles } \mathcal{E} \text{ of rank } n \text{ on } X \times S \\ \text{with minimal modification } \mathcal{E} \to (\mathrm{id} \times \mathrm{Frob}_S)^* \mathcal{E} \\ \text{at } r\text{-marked points } x_i : S \to X, 1 \leq i \leq r \end{array} \right\}$$

We have

$$\begin{array}{c} \mathrm{Sht}^r_{\mathrm{GL}_n,X} \\ \downarrow \\ X^r = \underbrace{X \times_{\mathrm{Spec} k} \cdots \times_{\mathrm{Spec} k} X}_{r \text{ times}} \end{array}$$

# Higher Gross–Zagier formula

## Theorem (Yun–Zhang)

*Fix $r \in \mathbb{Z}_{\geq 0}$. Let $E$ be a semistable elliptic curve over $k(X)$. Then there is an algebraic cycle (the* Heegner–Drinfeld cycle*) on $\mathrm{Sht}^r_{\mathrm{PGL}_2,X}$ such that the $E$-isotypic component of the cycle class $Z_{r,E}$ satisfies*

$$L^{(r)}(1,E) = c \cdot \left(Z_{r,E}, Z_{r,E}\right),$$

*where $(\cdot,\cdot)$ is the intersection pairing.*

The Heegner–Drinfeld cycle is defined analogous to Heegner point on modular curves, by imposing “complex multiplication": those vector bundles coming from a double covering of the curve $X$.

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# Comparison with the number field case

In the number field case, the analogous spaces only exist when $r \leq 1$.

1. When $r = 0$, this is the double-coset space

$$G(F) \backslash (G(\mathbb{A})/K) .$$

2. When $r = 1$, the analogous space is Shimura variety

$$\begin{array}{ccc} \mathrm{Sh}_G & & \mathrm{Sht}_{\mathrm{GL}_n, r} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbb{Z} & & X^r = \underbrace{X \times_{\mathrm{Spec}\,k} \cdots \times_{\mathrm{Spec}\,k} X}_{r \text{ times}} \end{array}$$

In the function field case, we need not restrict ourselves to the leading coefficient in the Taylor expansion of the $L$-functions.

**Question**

*In the number field case, should there be any geometric interpretation of the non-leading coefficients, for example, $L^{(r)}(1, E)$ when $E$ is an elliptic curve over $\mathbb{Q}$?*

Recall that the conjecture of Birch and Swinnerton-Dyer gives a geometric interpretation of the leading term

$$L^{(r)}(1, E) = c \cdot \mathrm{Reg}_E \cdot \mathrm{III}_E.$$

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# Intersection pairing on an algebraic surface

$S$: smooth projective surface over a field $k$.
$\mathrm{Div}(S)$: free abelian group of divisors on $S$.
There is an intersection pairing

$$\begin{aligned} \mathrm{Div}(S) \times \mathrm{Div}(S) &\to \mathbb{Z} \\ (C, D) &\mapsto C \cdot D \end{aligned}$$

# Hodge index theorem for a surface

**Theorem**

*Let $S$ be a surface over a field $k$.*

*If $H$ is an ample divisor, and $D \cdot H = 0$, then*

$$D \cdot D \leq 0.$$

$\mathrm{NS}(S) = \mathrm{Div}(S)$ modulo numerical equivalence. Then the index of the intersection matrices of a basis of $\mathrm{NS}(S)$ is

$$(+, -, -, -, \cdots).$$

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# Weil's proof of RH for curves

Consider a curve $X/\mathbb{F}_q$, and the surface

$$S = X \times_{\mathrm{Spec}\mathbb{F}_q} X$$

Compute the intersection matrix of 4 divisors

$$pt \times X, \quad X \times pt, \quad \Delta, \quad F$$

$F$ is the graph of the Frobenius

$$\mathrm{Frob}_q : X \to X.$$

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# Weil's proof of RH for curves

Denote $N = X(\mathbb{F}_q)$. The intersection matrix

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & q \\ 1 & 1 & 2-2g & N \\ 1 & q & N & q(2-2g) \end{pmatrix}$$

$$H = pt \times X + X \times pt \quad \text{ample}$$

$$\Longrightarrow det(T) = (N - (1+q))^2 - 4qg^2 \leq 0$$

$$\Longrightarrow |N - (1+q)| \leq 2g\sqrt{q}.$$

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# Arithmetic surface

An arithmetic surface $\overline{\mathcal{X}}$ is the data of a relative curve $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ with a metric on the Riemann surface $X(\mathbb{C})$.

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longrightarrow & \overline{\mathcal{X}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Q} & \longrightarrow & \mathrm{Spec}\mathbb{Z} & \longrightarrow & \overline{\mathrm{Spec}\mathbb{Z}} \end{array}$$

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# Hode index theorem for arithmetic surface

## Theorem (Faltings, Hriljac)

*Let $\overline{\mathcal{X}}$ be an arithmetic surface.*

*If $\overline{H}$ is an ample divisor, and $\overline{D} \cdot \overline{H} = 0$, then*

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## Remark

- This positivity together with Gross–Zagier formula implies $L'(1, E) \geq 0$. (in addition to RH over finite fields)
- Comparison the proof of $L'(1, E) \geq 0$ with the proof of RH for curve over a finite field. The geometric ingredients in them seem to be the best evidence to RH.

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# Yuan's proof of Hodge index for arithmetic surfaces

Yuan: an arithmetic line bundle $\overline{\mathcal{L}} \mapsto$ a convex body in $\mathbb{R}^2$.

**Lemma (Brunn–Minkowski)**

*Let $A, B$ be two compact subsets of $\mathbb{R}^n$, and let $A + B$ denote the Minkowski sum*

$$A + B = \{a + b : a \in A, b \in B\} \subset \mathbb{R}^n.$$

*Then*

$$\mathrm{vol}(A + B)^{1/n} \geq \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}.$$

# Surfaces

- The first kind is a surface over a filed $k$, e.g. $C \times C$ for a curve $C$ over $k$.
- The second kind is arithmetic surface: its base is an arithmetic curve $\mathrm{Spec}\mathbb{Z}$ and its fiber are curves over fields.
- The third kind is unknown: "$\mathrm{Spec}\mathbb{Z} \times_{\mathrm{Spec}\mathbb{F}_1} \mathrm{Spec}\mathbb{Z}$"? It should be a fibration with its base an arithmetic curve $\mathrm{Spec}\mathbb{Z}$ and with fibers also being arithmetic curves.
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# The third example: ABC and Landau–Seigel zeros

## Definition

A Landau–Siegel zero is a zero $\beta$ of $L(s, \chi_d)$ (for the quadratic character $\chi_d$ associated to $\mathbb{Q}[\sqrt{d}]$) lying in

$$[1 - c/\log|d|, 1]$$

for a small $c > 0$.

## Theorem (Granville–Stark)

*A uniform (over number fields) version of ABC conjecture implies that there are no Siegel zeros for $L(s, \chi_{-d})$ with $-d < 0$.*

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# Faltings heights of CM abelian varieties

The key to the theorem of Granville–Stark is Kronecker limit formula for an imaginary quadratic field $K = \mathbb{Q}[\sqrt{-d}]$. This formula relates the Faltings height of an elliptic curve $E_d$ with complex multiplication by $O_K$ to L-function

$$h_{\mathrm{Fal}}(E_d) = -\frac{L'(0,\chi_{-d})}{L(0,\chi_{-d})} - \frac{1}{2}\log|d|.$$

Colmez conjecture generalizes the identity to CM abelian varieties. An averaged version is recently proved by Yuan–S. Zhang and by Andreatta–Goren–Howard–Pera.

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