Integrable turbulence: A review of recent results in optics

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Nonlinear Statistical Optics: an historical perspective

Introduction

Lasers 1960s

Nonlinear optics
2\textsuperscript{nd} harmonic generation, Stimulated Raman/Brillouin scattering …

Statistical Optics/Optical Coherence Theory
Linear Theory (degree of coherence of light)

Optical fibers 1970s

Nonlinear fiber optics
Propagation of intense cw/pulsed coherent light waves in fibers

Telecommunications applications (linear operation)

Optical fibers 1995
Photonic crystal fibers
Management of fiber dispersion

Supercontinuum generation
High power/highly multimode lasers
Random fiber lasers…..

Nonlinear statistical optics
Supercontinuum generation in Photonic Crystal fibers

Source: University of Bath


« Wave Turbulence (WT) can be generally defined as out-of-equilibrium mechanics of random nonlinear waves »


In optics, WT theory has been applied in several circumstances to describe:

- *Spectral properties of random fiber lasers*, see e.g. Churkin et al, Nature Comm. 2, 6214 (2015)
- *Optical wave condensation*, see e.g. Connaughton et al, Phys. Rev. Lett. 95, 263901 (2005)

Supercontinuum generation is described by generalized 1D-NLS equations

\[ \frac{\partial A}{\partial z} = i \sum_{m \geq 2} \frac{i^m \beta_m}{m!} \frac{\partial^m A}{\partial t^m} + i \gamma \left[ 1 + i \tau_s \frac{\partial}{\partial t} \right] \times \left[ A(z, t) \int_{-\infty}^{+\infty} R(t') |A(z, t - t')|^2 dt' \right] - \hat{\alpha} A \]
WT treatment of supercontinuum generation in optical fibers

\[-i \frac{\partial \psi}{\partial z} = \sum_{j \geq 2} \frac{i^j \beta_j}{j!} \frac{\partial^j \psi}{\partial t^j} + \gamma |\psi|^2 \psi + i \gamma \tau_s \frac{\partial (|\psi|^2 \psi)}{\partial t}\]

Non-integrable PDE

\[\partial_z n(z, \omega_1) = C[n]\]

\[C[n] = \int n(\omega_1)n(\omega_2)n(\omega_3)n(\omega_4)[n^{-1}(\omega_1) + n^{-1}(\omega_2) - n^{-1}(\omega_3) - n^{-1}(\omega_4)] W d\omega_2 d\omega_3 d\omega_4,
\]

\[W = 2 \pi \gamma \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k(\omega_1) + k(\omega_2) - k(\omega_3) - k(\omega_4))\]

\[n^{eq}(\omega) = \frac{T(1 + \tau_s \omega)}{k(\omega) + \lambda \omega - \mu},\]

B. Barviau, B. Kibler and A. Picozzi
At leading order, the nonlinear propagation of waves in single-mode fibers is governed by the integrable 1D nonlinear Schrödinger equation:

\[
i \frac{\partial \psi}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 \psi}{\partial t^2} - \gamma |\psi|^2 \psi
\]

### Experiments with COHERENT initial conditions

Exact solutions (Peregrine solitons, collisions of Akhmediev Breathers...)

### Experiments in Optical Fibers

B. Kibler et al., Scientific Reports 2, 790, (2012)

### Experiments in 1D Water Tank

The integrable 1D nonlinear Schrödinger equation with random initial conditions and WT theory

\[ i \frac{\partial \psi}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 \psi}{\partial t^2} - \gamma |\psi|^2 \psi \]

Linear dispersion relation

\[ k(\omega) = \frac{\beta_2}{2} \omega^2 \]

\[ k(\omega_1) + k(\omega_2) = k(\omega_3) + k(\omega_4) \]
\[ \omega_1 + \omega_2 = \omega_3 + \omega_4 \]

\[ \omega_3 = \omega_{1,2} \]
\[ \omega_4 = \omega_{1,2} \]

\[ \frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{\gamma}{\pi} \int \int \int d\omega_{2-4} \mathcal{N}(z) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k(\omega_1) + k(\omega_2) - k(\omega_3) - k(\omega_4)) = 0 \]

\[ \mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_3}(z)n_{\omega_4}(z) + n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z) - n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z) - n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_4}(z) \]

Wave turbulence in Optics

Introduction

Integrable Turbulence

“Nonlinear wave systems integrable by Inverse Scattering Method (ISM) could demonstrate a complex behavior that demands the statistical description. The theory of this description composes a new chapter in the theory of wave turbulence-Turbulence in Integrable Systems”


Integrable Turbulence in optics

\[ i \frac{\partial \psi}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 \psi}{\partial t^2} - \gamma |\psi|^2 \psi \]

\[ \psi(z = 0, t) : \text{Complex random variable} \]
\[ i \psi_t + \psi_{xx} + 2 \sigma |\psi|^2 \psi = 0 \quad \sigma = \pm 1 \quad x \in [0, L] \]

Initial condition:
\[ \psi(x, t = 0) = \psi_0(x) = \sum_n \psi_{0n} e^{i k_0 x} \]

\[ k_0 = \frac{2 \pi}{L} \]

\[ \psi_{0n} = \frac{1}{L} \int_0^L \psi_0(x) e^{-i k_0 x} dx \]

\[ \phi_{on} \text{ randomly distributed between } -\pi \text{ and } \pi \]

Random Phase (RP) model

\[ R_0(x) = \Re(\psi_0(x)) \]

\[ P_0(x) = |\psi_0(x)|^2 \]

In optics, \( P_0 \) represents the wave power.

Gaussian statistics
Wave turbulence in Optics

Integrable Turbulence

Focusing regime

\[ i\psi_t + \psi_{xx} + 2\sigma |\psi|^2 \psi = 0 \]

Defocusing regime

|\psi(x,t)|^2

Heavy-tailed deviations from the initial gaussian statistics

Low-tailed deviations from the initial gaussian statistics
Another initial condition: plane wave + noise

\[ \psi(x, t = 0) = 1 + \eta(x) \quad |\eta(x)| \ll 1 \]

\( \eta(x) \) Complex random variable

The statistics is Gaussian at long evolution time.

Outline of the talk

- **Weakly nonlinear regime**: Wave Turbulence (WT) approach
- **Strongly nonlinear regime**: Dispersive hydrodynamics (semiclassical) approach
  - Defocusing regime (Observation of Riemann waves in integrable turbulence- Intermittency phenomenon)
  - Focusing regime (Fast measurement techniques/ Observation of Peregrine-like coherent structures in integrable turbulence)
  - Universality of the Peregrine Soliton in the Focusing Dynamics of the Cubic Nonlinear Schrödinger Equation
- Nonlinear spectral analysis of numerical and experimental signals
- **Conclusions / Future works**
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WT treatment of the propagation of weakly nonlinear random waves in 1D-NLSE systems

\[ i \partial_z \psi(z, t) = -\sigma \partial_t^2 \psi(z, t) + |\psi(z, t)|^2 \psi(z, t) \quad \sigma = \pm 1 \]

\[ \langle \bar{\psi}(z, \omega) \bar{\psi}^*(z, \omega') \rangle = n_\omega(z) \delta(\omega - \omega') \]

\[ \langle \bar{\psi}(z, \omega_1) \bar{\psi}(z, \omega_2) \bar{\psi}^*(z, \omega_3) \bar{\psi}^*(z, \omega_4) \rangle = J_{1,2}^{3,4}(z) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \]

\[ H_{NL} = \frac{1}{2} \int |\psi(z, t)|^4 dt \quad << \quad H_L = \int k(\omega) \bar{\psi}(z, \omega) d\omega \]

Weakly nonlinear regime (\( H_{NL} \ll H_L \)), nearly gaussian statistics

\[ \frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi} \int \int \int d\omega_2 d\omega_3 d\omega_4 \text{Im} \left[ J_{1,2}^{3,4}(z) \right] \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \]

\[ \frac{\partial J_{1,2}^{3,4}(z)}{\partial z} - i \Delta k J_{1,2}^{3,4}(z) = \frac{i}{\pi} \mathcal{N}(z) \]
WT Treatment of the propagation of weakly nonlinear random waves in 1D-NLSE systems

\[ \frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi^2} \int \int \int d\omega_2 d\omega_3 d\omega_4 N(z=0) \frac{\sin(\Delta k z)}{\Delta k} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \]

“Quasi-kinetic equations”

\[ \Delta k = k(\omega_1) + k(\omega_2) - k(\omega_3) - k(\omega_4) \]

Very weakly nonlinear regime

WT treatment relevant only to a (very) limited number of experiments in which the spectral broadening is very small


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Small dispersion NLS

\[ ie \frac{\partial \psi}{\partial \xi} + \frac{\epsilon^2}{2} \frac{\partial^2 \psi}{\partial \tau^2} - |\psi|^2 \psi = 0 \]

Defocusing propagation regime/Normal dispersion

\[ L_{NL} = 1/(\gamma \bar{\rho}_0) \quad \quad \quad L_D = 2/(\beta_2[\Delta \nu_0]^2) \]
\[ \beta_2 = +20 \text{ ps}^2 \text{ km}^{-1}, \gamma = +6 \text{ W}^{-1} \text{ km}^{-1} \]

\[ \epsilon = \sqrt{L_{NL}/L_D} \]

\[ \epsilon \ll 1 \quad \text{Strongly nonlinear regime} \]

\[ \psi(\tau, \xi) = \sqrt{\rho(\tau, \xi)} e^{i[\phi(\tau, \xi)/\epsilon]} \quad \text{Madelung Transformation} \]

Shallow water equations

In the pre-breaking regime

\[ \rho_{\xi} + (\rho u)_\tau = 0. \quad \quad \quad u(\tau, \xi) = \phi_{\tau} \]
\[ u_{\xi} + uu_{\tau} + \rho_{\tau} = 0 \]

\[ \rho: \text{fluid height/optical power} \]
\[ u: \text{depth-averaged horizontal velocity/instantaneous frequency} \]
Defocusing regime

\[ i \epsilon \frac{\partial \psi}{\partial \xi} + \frac{\epsilon^2}{2} \frac{\partial^2 \psi}{\partial \tau^2} - |\psi|^2 \psi = 0 \quad NLS \text{ equation} \]

\[ \rho \xi + (\rho u) \tau = 0, \quad \text{Shallow water equations} \]
\[ u \xi + uu \tau + \rho \tau = 0 \]

\[ r_{1,2}(\tau, \xi) = u \pm 2\sqrt{\rho} \quad \text{Riemann invariants} \]

\[ \frac{\partial r_{1,2}}{\partial \xi} + V_{1,2} \frac{\partial r_{1,2}}{\partial \tau} = 0 \quad \text{Propagation equations of two interacting Riemann waves} \]

\[ V_{1,2} = \frac{3}{4} r_{1,2} + \frac{1}{4} r_{2,1} \]

Pre-breaking dynamics
**Integrable turbulence**

**Defocusing regime/pre-breaking dynamics**

The initial statistics (at $\xi=0$) is gaussian for $\psi$

**Hydrodynamical variables $(\rho,U)$**

**Riemann invariants $(r_1,r_2)$**

\[
\rho(\tau,\xi) \approx \rho_0(\tau) + \frac{1}{4} [\rho_0^2(\tau)]'' \xi^2, \quad u(\tau,\xi) \approx -\rho_0'(\tau) \xi
\]

The PDF of the random Riemann wave field is invariant with respect to the $\xi$ evolution

Observation of optical Random Riemann waves in integrable turbulence

Defocusing regime

Observation of optical Random Riemann waves in integrable turbulence

Defocusing regime

Experimental results

Integrable turbulence in the Defocusing regime

Pre-breaking regime (random Riemann waves)

Post-breaking regime

Generation of dispersive shock waves (DSWs)

Influence of the interaction among DSWs on the statistics ??
Integrable Turbulence

Wave turbulence in Optics

Intermittency phenomenon

E. Falcon, S. Fauve, and C. Laroche,
“Observation of Intermittency in Wave Turbulence,

Wave turbulence in Optics

Integrable Turbulence

Intermittency phenomenon

Similar statistical features occur in NLS systems

Intermittency in integrable turbulence

S Randoux, P Walczak, M Onorato, P Suret

Physical review letters 113 (11), 113902 (2014)
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Theoretical challenge: statistical description of nonlinear random waves
A. Picozzi et al., Physics Reports, (2014)

Experimental challenge: sub-picosecond measurement of randomly-fluctuating light (irregular and non reproducible optical signal)

Experimental strategies/tools
- Asynchronous optical sampling (Characterization of the statistics of the field)
- Time Microscope (Characterization of the power fluctuations of the field)
- Heterodyne Time Microscope (Simultaneous characterization of the phase and the power fluctuations of the field)
Asynchronous Optical Sampling

Sum frequency generation signal + fs pulses

Temporal resolution: 250 fs

Wave turbulence in Optics

Fast measurement techniques

Equivalence Time / Space

Group velocity dispersion

\[ \frac{\partial \psi(z, t)}{\partial z} = i \frac{\beta_2}{2} \frac{\partial^2}{\partial t^2} \psi(z, t) \]

Time microscope

Sum Frequency Generation with chirped pulse

\[ A_{out}(x, y) = A_{in}(x, y) e^{i \alpha t} e^{-\frac{t^2}{T}} \]

1560 nm signal to be recorded

\[ \rightarrow \text{time to wavelength conversion} \]

G1, G2

irr filter

SFG signal (528 nm)

BBO crystal

grating (1800 l/mm)

200 mm focal length

camera focused at infinity

image to be recorded

\[ \rightarrow \text{space to angle conversion} \]

Propagation (paraxial approximation)

\[ \frac{\partial A(z, x, y)}{\partial z} = i \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} A(z, x, y) \]

\[ A_{out}(x, y) = A_{in}(x, y) e^{i \frac{x^2 + y^2}{2f}} \]

P. Suret et al., Nature Communications (2016)
Integrable Turbulence

✓ Time microscope

Random Light Source → Nonlinear Propagation (0.5 km fiber) → Time Microscope

\[ z = 0 \]
Initial Conditions
\[ \Delta \nu = 0.1 \text{ THz} \]

\[ z = 500 \text{ m} \]
\[ \langle P \rangle = 0.5 \text{ W} \]

\[ z = 500 \text{ m} \]
\[ \langle P \rangle = 2.6 \text{ W} \]

\[ z = 500 \text{ m} \]
\[ \langle P \rangle = 4. \text{ W} \]

P. Suret et al., Nature Communications (2016)
Wave turbulence in Optics

Integrable Turbulence

Fast measurement techniques

Experiments

Numerical simulations

Is it the Peregrine soliton?

Simultaneous measurement of power and phase: Heterodyne Time Microscope
Heterodyne Time microscope

- 800 nm chirped pump (dispersion $D_2>0$)
- 1560 nm signal to be recorded
- Dispersion $D_1<0$
- Camera focused at infinity
- 1560 nm reference
- $\chi^{(2)}$
- BBO
- 800 nm
- 1560 nm
- Fast measurement techniques
- Integrable Turbulence
Experimental observation of a $\pi$-phase jump confirming the emergence of Peregrine-like solitons in integrable turbulence

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\[ i\varepsilon \psi_t + \frac{1}{2} \varepsilon^2 \psi_{xx} + |\psi|^2 \psi = 0, \quad \varepsilon \ll 1. \]

For analytic initial data (gaussian or sech), dispersive regularization of a gradient catastrophe

- Finite lifespan of genus 0 solution: gradient catastrophe at some point \((x_0, t_0)\).
- For \(t > t_0\): a highly oscillatory (scaled as \(\varepsilon\)) well ordered region is formed, which is confined to a breaking curve.

The behaviour inside the breaking curve is described in terms of slowly modulated genus 2 solution (e.g. Kamvissis, McLaughlin and Miller (2003), Tovbis, Venakides and Zhou (2004)), outside the breaking curve it is a slowly modulated plane wave (genus 0).

![Figure: Evolution of \(e^{-x^2} \cosh \frac{i}{\varepsilon}, \varepsilon = 0.03\); Peregrine breather](image)

Bertola and Tovbis (2013): the universal behaviour (Dubrovin) near the point \(x_0, t_0\) of generic gradient catastrophe, in particular:

- The shape of each spike is universally described by the Peregrine breather solution, scaled to the size \(O(\varepsilon)\);
Numerical simulation of \[ i\varepsilon \frac{\partial \psi}{\partial \xi} + \frac{\varepsilon^2}{2} \frac{\partial^2 \psi}{\partial \tau^2} + |\psi|^2 \psi = 0 \] with \( \psi(\tau, \xi = 0) = \text{sech}(\tau) \quad \varepsilon = 0.02 \)

Peregrine soliton

**Integrable Turbulence**

 Universality of the peregrine soliton in the focusing dynamics of the cubic nonlinear Schrödinger equation

A Tikan et al, Physical review letters 119, 033901 (2017)
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Focusing NLS equation: IST integrability

The focusing NLS equation (NLSE)

\[ i \psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \]

where \( \psi(x, t) \) is a complex wave envelope. In the IST method, the NLSE is represented as the compatibility condition of two linear equations (Zakharov and Shabat 1972)

\[ Y_x = \begin{pmatrix} -i\lambda & \psi \\ \psi^* & i\lambda \end{pmatrix} Y, \]

\[ Y_t = \begin{pmatrix} -2i\lambda^2 + i|\psi|^2 & i\psi \psi_x + 2\lambda \psi \\ i\psi_x^* - 2\lambda \psi^* & 2i\lambda^2 - i|\psi|^2 \end{pmatrix} Y, \]

where \( \lambda \) is a complex spectral parameter and \( Y(x, t, \lambda) \) is a vector.

- The solution \( \psi(x, t) \) of the NLSE plays the role of the potential in the spectral problem (1) for the (non-selfadjoint) ZS operator.
- For decaying potentials, \( \psi \to 0 \) as \( |x| \to \infty \), the spectrum \( \{\lambda\} \) generally has two components: discrete and continuous.
- Discrete spectrum \( \rightarrow \) solitons.
  Continuous spectrum \( \rightarrow \) linear radiation.
- Isospectrality.
- Breathers? Need non-zero BCs (e.g. periodic).
FINITE-GAP POTENTIALS AND ROGUE WAVES

**General analytical framework** for the periodic NLSE: finite-gap theory
(*Its and Kotlyarov, 1976*)

Finite-gap (finite-band) NLS solutions are given by

\[
\psi_g(x, t) = \frac{\Theta_g(x, t; \nu^0_0)}{\Theta_g(x, t; \nu^0_+)} q e^{2i\eta^2 t},
\]

where \( \Theta_g \) is the Riemann theta-function of the hyperelliptic Riemann surface of genus \( g \) (Note: \( g = N - 1 \), where \( N \) is the number of bands),

\[
\Gamma_g : \quad \mathcal{R}(\lambda; \alpha, \bar{\alpha}) = \sqrt{(\lambda - \alpha_0)(\lambda - \bar{\alpha}_0) \cdots (\lambda - \alpha_g)(\lambda - \bar{\alpha}_g)},
\]

where \( \lambda \in \mathbb{C} \) is the spectral parameter and \( \alpha \in \mathbb{C}^{g+1} \) are the points of simple spectrum.

A genus \( g \) solution is a \( g \)-phase solution:

\[
\psi_g(x, t) = \psi_g(\eta_1, \ldots, \eta_g), \quad \text{where} \quad \eta_j = k_j x + \omega_j t + \eta_j^0.
\]

Here \( k_j = k_j(\alpha, \bar{\alpha}), \omega_j = \omega_j(\alpha, \bar{\alpha}), j = 1, \ldots, g \) are given by certain hyperelliptic integrals on \( \Gamma_g \).

\[
\eta \in \mathbb{T}^g : \quad \psi_g(\ldots, \eta_j + 2\pi, \ldots) = \psi_g(\ldots, \eta_j, \ldots) \text{ for all } j = 1, \ldots, g.
\]

Hence \( \psi_g(x, t) \) is quasi-periodic in both \( x \) and \( t \) provided \( k_j \) and \( \omega_j \) are incommensurable.
**Solitons and Breathers: Spectral Portraits (Periodic IST)**

Genus 0 solution: (slowly-modulated) plane waves

Genus 1 solution: cnoidal waves

Two bands

Genus 2 solution: solitons on finite background

Three bands

Akhmediev Breather

Peregrine soliton

Kuznetsov-Ma soliton
Noise driven modulational instability: local periodic ST analysis \( \psi(x, t = 0) = 1 + \eta(x) \)

“Inverse scattering transform analysis of rogue waves using local periodization procedure”

Nonlinear spectral analysis of the PS recorded in the hydrodynamical experiment by Chabchoub et al, PRL 106, 204502 (2011)

S. Randoux, P. Suret, A. Chabchoub, B. Kibler, G. El
Manuscript in preparation
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Dispersive hydrodynamics is the domain of applied mathematics and physics concerned with fluid motion in which the internal friction, e.g., viscosity, is negligible relative to wave dispersion.

Dispersive hydrodynamics represents a very relevant framework for the analysis of integrable turbulence
- In the defocusing regime, the statistics of the random wave field before wave breaking can be described in terms of Riemann Waves. The PDF of the random Riemann wave field is invariant with respect to the $\xi$ evolution (contrary to the hydrodynamical variables that exhibit significant statistical changes).

- In the focusing regime, significant recent experimental progress have shown that the Peregrine soliton represents a coherent structure of fundamental importance in integrable turbulence.

- Nonlinear spectral analysis represents a tool of significant interest for the characterization of coherent structures emerging in integrable turbulence.
Open questions / Future works

- Connection between Fourier (linear) spectrum and IST (nonlinear) spectrum
- Statistical theory of integrable turbulence in the post-breaking regime
- Role of pertubative effects (dissipation…)
- Soliton gas in Optics and in Hydrodynamics
Thank you for your attention!
One of the key results of [Bertola and Tovbis, Comm. Pure Appl. Math. 66, 678 (2013)] is that when $\varepsilon \ll 1$ ($N \gg 1$) the dynamics near the gradient catastrophe universally leads to the generation of a rational PS as a local asymptotic solution of the NLSE.