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Quasi-invariant gaussian measures for nonlinear wave equations

joint work with Tadahiro Oh

The Hamiltonian structure

- Consider the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + u^3 = 0, \tag{1}$$

where $u : \mathbb{R} \times \mathbb{T}^d \longrightarrow \mathbb{R}$, $d \geq 1$.

- We rewrite (1) as the first order system

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{2}$$

- One can rewrite (2) as a Hamiltonian system

$$\partial_t u = \frac{\delta E}{\delta v}, \quad \partial_t v = -\frac{\delta E}{\delta u},$$

where

$$E(u,v) = \frac{1}{2}\int_{\mathbb{T}^d} \left(u^2 + |\nabla u|^2 + v^2\right) + \frac{1}{4}\int_{\mathbb{T}^d} u^4 .$$

- Therefore $E(u,v)$ is a first integral for (2).

The global well-posedness for $d \leq 3$

• In view of the Hamiltonian structure and the properties of the linear equation, a natural phase space for

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{3}$$

is $\mathcal{H}^s(\mathbb{T}^d) \equiv H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$.

Theorem 1 (classical)

Let $d \leq 3$. Then (3) is globally well-posed in $\mathcal{H}^s(\mathbb{T}^d)$, $s \geq 1$. More precisely, for every $(u_0, v_0) \in \mathcal{H}^s(\mathbb{T}^d)$ there is a unique solution of (3) in $C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^d))$.

Remark. The result still holds true for $d = 4$ by the concentration compactness method. For $d > 5$ the problem has some similarities with the 3d Navier-Stokes global regularity problem.

• Denote by $\Phi(t) : \mathcal{H}^s(\mathbb{T}^d) \to \mathcal{H}^s(\mathbb{T}^d)$ the resulting flow in Theorem 1.

• **In this lecture, we are interested in the statistical description of the flow $\Phi(t)$.**

Formal definition of the gaussian measures

- Let $\mu_{s,d}$ be the measure **formally** defined as

$$d\mu_{s,d} = Z_{s,d}^{-1} e^{-\frac{1}{2}\|(u,v)\|^2_{\mathcal{H}^{s+1}}} du dv$$

or

$$\prod_{n \in \mathbb{Z}^d} Z_{s,d,n}^{-1} \; e^{-\frac{1}{2}\langle n\rangle^{2(s+1)}|\widehat{u}_n|^2} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{v}_n|^2} d\widehat{u}_n d\widehat{v}_n \,,$$

where $\widehat{u}_n$ and $\widehat{v}_n$ denote the Fourier transforms of u and v respectively.

- Notation : $\langle n\rangle = (1+|n|^2)^{\frac{1}{2}}$.

Rigorous definition of the gaussian measures

• $\mu_{s,d}$ is the induced probability measure under the map

$$\omega \longmapsto (u^\omega(x), v^\omega(x))$$

with

$$u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^{s+1}} e^{in\cdot x}, \qquad v^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{h_n(\omega)}{\langle n\rangle^{s}} e^{in\cdot x} . \tag{4}$$

• In (4), $(g_n)_{n\in\mathbb{Z}^d}$, $(h_n)_{n\in\mathbb{Z}^d}$ are two sequences of "standard" complex gaussian random variables, such that $g_n = \overline{g_{-n}}$, $h_n = \overline{h_{-n}}$ and such that $\{g_n, h_n\}$ are independent, modulo the central symmetry.

• The partial sums of the series in (4) are a Cauchy sequence in $L^2(\Omega; \mathcal{H}^\sigma(\mathbb{T}^d))$ for every $\sigma < s+1-\frac{d}{2}$ and therefore one can see $\mu_{s,d}$ as a probability measure on $\mathcal{H}^\sigma$ for a fixed $\sigma < s+1-\frac{d}{2}$.

Remark. For the same range of σ, the triplet $\left(\mathcal{H}^{s+1}(\mathbb{T}^d), \mathcal{H}^\sigma(\mathbb{T}^d), \mu_{s,d}\right)$ forms an abstract Wiener space.

Statement of the results

• By the global well-posedness result, we also have that the flow $\Phi(t)$ is defined $\mu_{s,2}$ almost surely, provided $s > 1$.

Theorem 2
Let $s \geq 2$ be an even integer. Then $\mu_{s,2}$ is quasi-invariant under $\Phi(t)$.

• In $1d$ one can show the quasi-invariance of $\mu_{s,1}$ for $s \geq 0$ with a much simpler proof.
• The result of Theorem 2 holds also for the cubic wave equation

$$\partial_t^2 u - \Delta u + u^3 = 0 \, .$$

A corollary

• Denote by $u(t,x,\omega)$ the solution of the nonlinear wave equation with data

$$u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^{s+1}} e^{in\cdot x}, \qquad v^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{h_n(\omega)}{\langle n\rangle^{s}} e^{in\cdot x} .$$

• Then there exist $f(t,\omega) \geq 0$, $f(t,\cdot) \in L^1(\Omega)$, $f(0,\omega) = 1$, such that we have the conservation laws :

$$\int_\Omega \widehat{u}_n(t,\omega)\overline{\widehat{u}_m(t,\omega)} f(t,\omega) dp(\omega) = 0, \qquad n \neq m,$$

and

$$\int_\Omega |\widehat{u}_n(t,\omega)|^2 f(t,\omega) dp(\omega) = \langle n\rangle^{-2(s+1)} ,$$

where $u_n(t,\omega)$ are the Fourier coefficients of $u(t,x,\omega)$.

• It might be that the densities $f(t,\omega)$ should be taken into account in wave turbulence considerations. This remark also apples to other models when one has quasi-invariant measures.

Comments

• The proof crucially exploits the "dispersion" for any s. More precisely, it is likely that $\mu_{s,2}$ is not quasi-invariant under the flow of

$$\partial_t u = v, \quad \partial_t v = -u - u^3.$$

• We expect that the same result should hold for any $s > 0$ (for $s \in (0,1]$ one should use a probabilistic global well-posedness in the sense of Burq-Tz.).

• We have some hope to extend the results to $d = 3$ ...

• Main remaining issue : What can be said about the resulting densities ?

Comments (sequel)

• When writing the wave equation as a first order system there is a one smoothing in the nonlinearity. This fact is crucially exploited in the proof.
• One can however extend our methods and results to models such that there is no smoothing in the nonlinearity as the fourth order NLS

$$i\partial_t u + \partial_x^4 u = |u|^2 u.$$

The proof in this case requires the use of **normal form and gauge transformations**.

Comments (sequel)

• By using **the integrability** (the Lax pair structure) one may prove similar quasi-invariance results for

$$i\partial_t u + \partial_x^2 u = |u|^2 u.$$

• But we do not know presently what happens for

$$i\partial_t u + \partial_x^2 u = |u|^4 u$$

or

$$i\partial_t u + \Delta u = |u|^2 u.$$

in two dimensions.

From now on we only consider $d = 2$ **and we denote** $\mu_{s,2}$ **simply by** μ_s.

Related results 1 (Cameron-Martin 1944)

Theorem 3 (CM in the context of the measure μ_s)
For a fixed $(h_1, h_2) \in \mathcal{H}^\sigma$, $\sigma < s$, the transport of μ_s under the shift

$$(u_1, u_2) \longmapsto (u_1, u_2) + (h_1, h_2),$$

is absolutely continuous with respect to μ_s if and only if

$$(h_1, h_2) \in \mathcal{H}^{s+1}\,.$$

Proof. Denote by $(\cdot,\cdot)$ the scalar product in $\mathcal{H}^{s+1}(\mathbb{T}^2)$.
Let first $h = (h_1, h_2) \in \mathcal{H}^{s+1}$. If $u = (u_1, u_2) \in \mathcal{H}^{s+1}(\mathbb{T}^2)$ then

$$\|u + h\|^2_{\mathcal{H}^{s+1}} = \|u\|^2_{\mathcal{H}^{s+1}} + \|h\|^2_{\mathcal{H}^{s+1}} + 2(u, h).$$

Therefore, at least "formally" the transported measure is

$$e^{-\frac{1}{2}\|h\|^2_{\mathcal{H}^{s+1}}}\, e^{-(u,h)} d\mu_s(u)\,.$$

Proof of the Cameron-Martin theorem

• We are therefore reduced to check that μ_s almost surely the scalar product (u,h) is finite.
• Let us now observe that the proof of the fact that the scalar product (u,h) is μ_s almost surely finite reduces to check that for $(c_n) \in l^2(\mathbb{Z}^2)$,

$$\sum_{n\in\mathbb{Z}^2} c_n g_n(\omega) < \infty \quad \text{a.s} \quad \text{in} \quad \omega$$

which holds thanks to an orthogonality property resulting from the independence.
• Let now $h \notin \mathcal{H}^{s+1}$. Then there exists $g \in \mathcal{H}^{s+1}$ such that $(g,h) = \infty$.
• Define $A := \{w\,:\,(w,g) < \infty\}$. Then as above $\mu_s(A) = 1$.
• Denote by ρ_s the image measure of μ_s under the map $u \mapsto u + h$. Then

$$\rho_s(A) = \mu_s(B), \qquad B = \{w - h, w \in A\}.$$

Hence for every $u \in B$, $(u,g) = \infty$. Therefore $B \subset A^c$ and $\mu_s(B) = 0$. This in turn implies that $\rho_s(A) = 0$.
This completes the proof.

"Comparing" our result Cameron-Martin's theorem

• Let $S(t)$ be the free wave evolution.
• For $(u,v) \in \mathcal{H}^\sigma$, we classically have

$$\Phi(t)(u,v) = S(t)\Big((u,v) + (h_1, h_2)\Big),$$

where $(h_1,h_2) = (h_1(u,v), h_2(u,v)) \in \mathcal{H}^{\sigma+1}$ (one smoothing and not more).
• If $\sigma < s$ then $\sigma + 1 < s + 1$ and therefore our result displays a remarkable property of the vector field generating $\Phi(t)$.

Related results 2 (Ramer 1974)

• For $\sigma < s$, let us consider a diffeo Φ on $\mathcal{H}^{\sigma}(\mathbb{T}^2)$ of the form

$$\Phi(u,v) = (u,v) + F(u,v),$$

where $F : \mathcal{H}^{\sigma}(\mathbb{T}^2) \to \mathcal{H}^{s+1}(\mathbb{T}^2)$. Suppose that

$$DF(u,v) : \mathcal{H}^{s+1}(\mathbb{T}^2) \to \mathcal{H}^{s+1}(\mathbb{T}^2)$$

is Hilbert-Schmidt.

• Ramer (1974) : under the above assumption μ_s is quasi-invariant under Φ.

• Typical example :

$$F(u,v) = \varepsilon(1-\Delta)^{-1-\delta}(u^2,v^2), \quad \delta > 0,\ |\varepsilon| \ll 1,$$

i.e. 2-smoothing is needed.

• The Ramer's result would apply in the context of

$$\partial_t^2 u + (-\Delta)^{\alpha} u + u + u^3 = 0, \quad \alpha > 2.$$

• Therefore our result *seems* to go much beyond Ramer's framework because for the wave equation there is only 1-smoothing.

Related results 3. (A.B. Cruzeiro 1983)

• In her work Ana Bela Cruzeiro considers a general equation of the form

$$\partial_t u = X(u),$$

where X is a vector field on $\mathcal{H}^\sigma$, $\sigma < s$.

• A.B. Cruzeiro 1983 : the resulting flow has μ_s as a quasi-invariant measure provided that several assumptions are satisfied, the most important being

$$\int_{\mathcal{H}^\sigma} e^{\mathrm{div}(X(u))} d\mu_s(u) < \infty. \tag{5}$$

• Very roughly speaking, our work consists in verifying in practice a condition of type (5).

The approximated model

- We consider the approximated models

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3), \quad N \to \infty, \tag{6}$$

where π_N denotes the Dirichlet projector on Fourier modes $\leq N$, i.e.

$$(\pi_N u)(x) = \sum_{|n|\leq N} \widehat{u}(n)\, e^{in\cdot x}\,.$$

- The quantity

$$E_N(u,v) = \frac{1}{2}\int_{\mathbb{T}^2}\Big(u^2 + |\nabla u|^2 + v^2\Big) + \frac{1}{4}\int_{\mathbb{T}^2}(\pi_N u)^4$$

is conserved under the flow of (6).

- We can therefore obtain that as is the case of the non-truncated model, the Cauchy problem for (6) is still globally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$, $s \geq 1$.
- For shortness, in the sequel we denote $\pi_N u$ by u_N and $\pi_N v$ by v_N.

The generalised energies

• Taking into account the definition of the gaussian measure μ_s, it is natural to study the expression

$$\frac{1}{2}\frac{d}{dt}\|(u_N(t),v_N(t))\|^2_{\mathcal{H}^{s+1}},$$

where (u,v) is a solution of

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3)\,. \tag{7}$$

• For that purpose, we observe that if (u,v) is a solution of (7) then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u_N(t),v_N(t))\|^2_{\mathcal{H}^{s+1}} &= \partial_t\Big[\frac{1}{2}\int_{\mathbb{T}^2}(J^s v_N)^2+\frac{1}{2}\int_{\mathbb{T}^2}(J^{s+1}u_N)^2\Big] \\ &= \int_{\mathbb{T}^2}(J^{2s}v_N)(-u_N^3)\,,\end{aligned}$$

where $J\equiv\sqrt{1-\Delta}$. Of course, if $s=0$, the term in the r.h.s is

$$-\frac{1}{4}\partial_t\Big[\int_{\mathbb{T}^2}u_N^4\Big]$$

and we recover the conservation of $E(u_N,v_N)$.

The generalised energies (sequel)

- For $s \geq 2$, an even integer, using the Leibniz rule, we can write

$$\int_{\mathbb{T}^2} (J^{2s} v_N)(-u_N^3) = -3 \int_{\mathbb{T}^2} J^s v_N \, J^s u_N \, u_N^2 + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^2} J^s v_N \, \partial^\alpha u_N \partial^\beta u_N \partial^\gamma u_N,$$

where $c_{\alpha,\beta,\gamma}$ are unessential constants.

- Let us analyse the quantity

$$-3 \int_{\mathbb{T}^2} J^s v_N \, J^s u_N \, u_N^2 \, .$$

A need of a renormalisation

• Recalling that $\partial_t u_N = v_N$, we can write

$$
\begin{aligned}
-3\int_{\mathbb{T}^2} J^s v_N\, J^s u_N\, u_N^2 &= -\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2 u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\, v_N\, u_N \\
&= -\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}\mathbf{P}_{\neq 0}[(J^s u_N)^2]\,\mathbf{P}_{\neq 0}[u_N^2]\Big] \\
&\quad + 3\int_{\mathbb{T}^2}\mathbf{P}_{\neq 0}[(J^s u_N)^2]\,\mathbf{P}_{\neq 0}[v_N\, u_N] \\
&\quad - \frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2\int_{\mathbb{T}^2} u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\int_{\mathbb{T}^2} v_N\, u_N,
\end{aligned}
$$

where $\mathbf{P}_{\neq 0}$ is the projection on the non zero frequencies.

• The last two terms on the right-hand side are problematic because

$$\lim_{N\to\infty}\mathbb{E}_{\mu_s}\Big[\int_{\mathbb{T}^2}(J^s\pi_N u)^2\Big] = \infty\,.$$

Therefore, we need to use a suitable renormalisation !

The renormalised energies

- Define σ_N by

$$\sigma_N = \mathbb{E}_{\mu_s}\Big[\int_{\mathbb{T}^2}(J^s\pi_N u)^2\Big] = \sum_{\substack{n\in\mathbb{Z}^2\\|n|\leq N}} \frac{1}{1+|n|^2} \sim \log N\,.$$

- Then, we have

$$-\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2\int u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\int v_N\, u_N$$
$$= -\frac{3}{2}\partial_t\Big[\Big(\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N\Big)\int_{\mathbb{T}^2} u_N^2\Big] + 3\Big(\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N\Big)\int v_N\, u_N.$$

- The term

$$\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N$$

is now a "good" term because there is C such that for every $p \geq 2$ and every $N \geq 1$

$$\Big\|\int_{\mathbb{T}^2}(J^s\pi_N u)^2 - \sigma_N\Big\|_{L^p(d\mu_s(u,v))} \leq C\sqrt{p}\,.$$

The renormalised energies (sequel)

• In view of the above discussion, it is now natural to define the modified energy $E_{s,N}(u,v)$ by

$$E_{s,N}(u,v) = \frac{1}{2}\int (J^s v)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\sigma_N \int u^2$$

and we have that if that if (u,v) is a solution of the truncated problem then

$$\partial_t E_{s,N}(u_N, v_N) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s u_N)^2]\, \mathbf{P}_{\neq 0}[v_N\, u_N] + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^2} J^s v_N\, \partial^\alpha u_N \partial^\beta u_N \partial^\gamma u_N + 3\Big(\int_{\mathbb{T}^2} (J^s u_N)^2 - \sigma_N\Big)\int_{\mathbb{T}^2} v_N\, u_N. \quad (8)$$

• Now all terms in the right hand-side of (8) are suitable for a perturbative analysis.

The key estimate

Theorem 4

Let $s \geq 2$ be an even integer and let us denote by $\Phi_N(t)$ the flow of

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3)\,.$$

Then for every $r > 0$ there is a constant C such that for every $p \geq 2$ and every $N \geq 1$,

$$\Big(\int_{E_N(u,v) \leq r} \Big| \partial_t E_{s,N}(\pi_N \Phi_N(t)(u,v))|_{t=0} \Big|^p d\mu_s(u,v) \Big)^{\frac{1}{p}} \leq Cp.$$

On the proof of the quasi-invariance

• Recall that

$$E_{s,N}(u,v) = \frac{1}{2}\int (J^s v)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\sigma_N \int u^2$$

• By classical arguments from QFT, we can define

$$\lim_{N\to\infty}\left(\frac{3}{2}\int (J^s \pi_N u)^2(\pi_N u)^2 - \frac{3}{2}\sigma_N \int (\pi_N u)^2\right)$$

in $L^p(d\mu_s(u,v))$, $p < \infty$.

• Denote this limit by $R(u)$. Essentially speaking, once we have the key estimate, we study the quasi-invariance of

$$\mathbf{1}_{E(u,v)\leq r}\, e^{-R(u)}\, d\mu_s(u,v)$$

by soft analysis techniques.

On the proof of the quasi-invariance (sequel)

• Let us be a little more precise. Denote by $x(t)$ the measure evolution of a set having zero measure with respect to

$$\mathbf{1}_{E(u,v)\leq r}\, e^{-R(u)}\, d\mu_s(u,v)\,.$$

• Essentially speaking, using the key estimate and the Liouville argument, we obtain that $x(t)$ satisfy the estimate

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}, \quad x(0) = 0\,. \tag{9}$$

Integrating the last estimate leads to $x(t) \leq (Ct)^p$. Taking the limit $p \to \infty$, we infer that $x(t) = 0$ for $0 \leq t < 1/C$. Since C is an absolute constant, we can iterate the argument and show that $x(t)$ is vanishing.

• Observe that this argument would not work if in (9), we have p^α, $\alpha > 1$ instead of p.

• In order to make the previous reasoning rigorous, we need to use some more or less standard approximation arguments.

On the proof of the key proposition

- We have that

$$\partial_t E_{s,N}(\pi_N \Phi_N(t)(u,v))|_{t=0} = Q_1(u,v) + Q_2(u,v) + Q_3(u,v),$$

where

$$Q_1(u,v) = 3 \int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\, \mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u],$$

$$Q_2(u,v) = \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^2} J^s \pi_N v\, \partial^\alpha \pi_N u \partial^\beta \pi_N u \partial^\gamma \pi_N u,$$

$$Q_3(u,v) = 3 \Big(\int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big) \int_{\mathbb{T}^2} \pi_N v\, \pi_N u.$$

Estimate of $Q_3(u,v)$

- For $r>0$, we define $\mu_{s,r}$ as

$$d\mu_{s,r}(u,v) = \mathbf{1}_{E(u,v)\le r}\, d\mu_s(u,v)\,.$$

- The goal is to show that

$$\|Q_3(u,v)\|_{L^p(d\mu_{s,r}(u,v))} \le Cp,$$

with a constant C independent of N and p. Since

$$\left|\int_{\mathbb{T}^2} \pi_N v\, \pi_N u\right| \le \|\pi_N u\|_{L^2}\, \|\pi_N v\|_{L^2} \le E(u,v),$$

we obtain that

$$\begin{aligned}\|Q_3(u,v)\|_{L^p(d\mu_{s,r}(u,v))} &\le C_r \left\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \right\|_{L^p(d\mu_{s,r}(u,v))} \\ &\le C_r \left\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \right\|_{L^p(d\mu_s(u,v))}\,.\end{aligned}$$

Estimate of $Q_3(u,v)$ (sequel)

- On the other hand

$$\Big\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big\|_{L^p(d\mu_s(u,v))} = \Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^p(\Omega)}$$

and by using Wiener chaos estimates, we have

$$\Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^p(\Omega)} \le Cp \Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^2(\Omega)} \le Cp$$

which provides the needed bound for $Q_3(u,v)$.

On the estimate of $Q_1(u,v)$

• The bound for

$$Q_1(u,v) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\,\mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u]$$

is the most delicate part of the analysis and relies on subtle multi-linear arguments.

• Basically, we are allowed to have outputs as

$$\|J^\sigma u\|_{L^\infty(\mathbb{T}^2)}, \quad \sigma < s$$

with a loss $\sqrt{p}$ and $E(u,v)$.

• A naive Hölder inequality approach clearly fails.

• A purely probabilistic argument based on Wiener chaos estimates fails because the output power of p is too large.

• The basic strategy is to perform a multi-scale analysis redistributing properly the derivative losses by never having more then quadratic weight of the contribution of the Wiener chaos estimate.

On the estimate of $Q_1(u,v)$ (sequel)

• When analysing the 4-linear expression

$$Q_1(u,v) = 3 \int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\, \mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u],$$

we suppose that

$$J^s \pi_N u,\ J^s \pi_N u,\ \pi_N v,\ \pi_N u$$

are localised at frequencies N_1, N_2, N_3, N_4 respectively.

• We first consider the case when $N_4 \gtrsim (\max(N_1, N_2))^{\frac{1}{100}}$. In this case we exchange some regularity of $J^s \pi_N u$ with this of $\pi_N u$ and we perform the naive linear analysis.

On the estimate of $Q_1(u,v)$ (sequel)

• Therefore, in the analysis of

$$Q_1(u,v) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\, \mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u],$$

we can suppose that $N_4 \ll (\max(N_1, N_2))^{\frac{1}{100}}$. In this case, we have that

$$\max(N_1, N_2) \sim \max(N_j,\, j = 1,2,3,4).$$

• By symmetry, we can suppose that $N_1 = \max(N_1, N_2)$.
• We next consider the case $N_3 \ll N_1^{1-a}$, $a = a(s)$-small. In this case, we perform a bi-linear Wiener chaos estimate and we have **some gain of regularity** in the localisation of $\mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]$.

On the estimate of $Q_1(u,v)$ (sequel)

• Finally, we consider the case

$$N_1 \sim \max(N_j,\, j=1,2,3,4),\ N_4 \ll (\max(N_1,N_2))^{\frac{1}{100}},\ N_3 \gtrsim N_1^{1-a}$$

In this case, we perform a tri-linear Wiener chaos estimate and we have **enough gain of regularity** in the localisation of

$$\mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\pi_N v\,.$$

• This essentially explains the argument leading to the key estimate.