Ergodicity of the Gibbs measure for the three dimensional stochastic cubic beam equation with damping

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Stochastic Damped Beam Equation

We want to study the global behaviour of the equation

(SDNLB)
$$\begin{cases} \partial_t^2 u + \partial_t u + (1+\Delta^2)u + u^3 = \sqrt{2}\xi \\ u(0) = u_0 \\ \partial_t u(0) = u_1 \end{cases}$$

posed on $\mathbb{T}^3$, where ξ is the space-time white noise.

Vectorial formulation:

(SDNLB)
$$\begin{cases} \partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u_t \\ -(1+\Delta^2)u - u^3 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_t + \sqrt{2}\xi \end{pmatrix} \\ \begin{pmatrix} u \\ u_t \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

White Noise

The space-time white noise is a distribution-valued random process such that for every test function ϕ, $\langle \xi, \phi \rangle$ is a Gaussian variable,

$$\mathbb{E}\langle \xi, \phi \rangle = 0, \qquad \mathbb{E}|\langle \xi, \phi \rangle|^2 = \int_{\mathbb{R}\times\mathbb{T}^3} |\phi(t,x)|^2 \mathrm{d}t\mathrm{d}x.$$

From these properties,

$$\xi = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}^3} e^{inx} \mathrm{d}W_n(t),$$

where W_n are independent copies of the Brownian motion on $\mathbb{R}$.

The white noise is very rough: $\xi \in C_t^{-\frac{1}{2}-} H^{-\frac{3}{2}-}$

Invariant Measures

To study the long time behaviour of the flow, we look for *invariant* measures.

Definition:

Let $\Phi_t(u)$ be the flow of the equation.
A measure ρ is **invariant** if for every F bounded borel function, $\forall t \geqslant 0$,

$$(*) \qquad \int \mathbb{E}[F(\Phi_t(u))]\mathrm{d}\rho(u) = \int F(u)\mathrm{d}\rho(u).$$

Since $\mathbb{E}[F(\Phi_0(u))] = \mathbb{E}[F(u)] = F(u)$, $(*) \iff$

$$\int \mathbb{E}[F(\Phi_t(u))]\mathrm{d}\rho(u)$$

invariant in time.

Gibbs Measure

Recall the splitting

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u_t \\ -(1+\Delta^2)u - u^3 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_t + \sqrt{2}\xi \end{pmatrix}.$$

The equation $\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u_t \\ -(1+\Delta^2)u - u^3 \end{pmatrix}$ is Hamiltionian,

$$H(u, u_t) = \frac{1}{2}\int |u_t|^2 + \frac{1}{2}\int u^2 + \frac{1}{2}\int |\Delta u|^2 + \frac{1}{4}\int u^4.$$

$\Rightarrow$ expect H is conserved.

Liouville's theorem $\Rightarrow$ expect "$\mathrm{d}u\mathrm{d}u_t$" invariant

$\Rightarrow$ expect Gibbs measure $\rho =$ "$A^{-1}\exp(-H(u, u_t))\mathrm{d}u\mathrm{d}u_t$" invariant.

Gibbs measure

Second piece: $\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ -u_t + \sqrt{2}\xi \end{pmatrix}$.

It is constant in u and the *Orstein - Uhlenbeck* process in u_t:

$$\partial_t u_t = -u_t + \sqrt{2}\xi,$$

which is well known to have the invariant measure

$$\text{“}\exp\Big(-\frac{1}{2}\int |u_t|^2\Big)\mathrm{d}u_t\text{”}.$$

u constant on the flow $\Rightarrow$ expect "$F(u)\exp\Big(-\frac{1}{2}\int |u_t|^2\Big)\mathrm{d}u\mathrm{d}u_t$" invariant.

$\Rightarrow$ expect $\rho =$ "$A^{-1}\exp(-H(u,u_t))\mathrm{d}u\mathrm{d}u_t$" invariant for this flow as well.

$\Rightarrow$ expect $\rho =$ "$A^{-1}\exp(-H(u,u_t))\mathrm{d}u\mathrm{d}u_t$" invariant for Φ.

LWP

Mild formulation/variation of constants:

If $S(t)(u_0, u_1)$ solves $\partial_t^2 u + \partial_t u + (1+\Delta^2)u = 0$,

$$u = S(t)(u_0, u_1) + \int_0^t S(t-s)(0, \xi(s))\mathrm{d}s + \int_0^t S(t-s)(0, u^3(s))\mathrm{d}s,$$

$$u = Z + \psi + v.$$

$S(t)$ gains 2 derivatives $\Rightarrow$ we expect $\psi \in C_t H^{0-}$.
Stochastic integral $\Rightarrow$ $\psi \in C_t H^{1/2-}$. Moreover,

$$\psi \in C_t C_x^{\frac{1}{2}-}.$$

Equation for v:

$$v = \int_0^t S(t-s)(0, (v + \psi + Z)^3)\mathrm{d}s.$$

Banach Fixed point argument $\Rightarrow$ LWP for $v \in H^2$, as long as

Need: $$Z \in C_{t,x}.$$

GWP

Consider the energy

$$E(v) = \frac{1}{2}\int |v_t|^2 + \frac{1}{2}\int v^2 + \frac{1}{2}\int |\Delta v|^2 + \frac{1}{4}\int v^4.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(v) &= \int \partial_t v\big[\partial_t^2 v + v + \Delta^2 v + v^3\big] \\
&= \int \partial_t v\big[v^3 - (v + \psi + Z)^3\big] \\
&= -\int (\partial_t v)^2 + 3\int (\partial_t v)(v^2)(\psi + Z) + \text{l.o.t.} \\
&\lesssim 1 + E^{\frac{1}{2}} E^{\frac{1}{2}} (\|\psi\|_{L^\infty} \|Z\|_{L^\infty}) \\
&= 1 + E(\|\psi\|_{L^\infty} \|Z\|_{L^\infty})
\end{aligned}$$

Gronwall $\Rightarrow$ Global Existence.
Remark: Differentiating $E + \varepsilon \int (v + \partial_t v)^2$, we get that the solution is uniformly bounded in time (instead of getting exponential growth).

Invariance

How to prove invariance? Galerkin approximations.
Let P_N be the projection on $\mathrm{Span}\{e^{inx}, |n| \leqslant N\}$. Consider

$$\begin{cases} \partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u_t \\ -(1+\Delta^2)u - P_N u^3 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_t + \sqrt{2} P_N \xi \end{pmatrix} \\ \begin{pmatrix} u \\ u_t \end{pmatrix}(0) = \begin{pmatrix} P_N u_0 \\ P_N u_1 \end{pmatrix} \end{cases}$$

This is a finite dimensional system of *SDEs*, for which we expect

$$\rho_N := A_N^{-1} \exp(-H(P_N u, P_N u_t))\mathrm{d}u\mathrm{d}u_t$$

to be invariant.
Finite dimension $\Rightarrow$
invariance $\iff$ $\exp(-P_N H(u, u_t))$ solves the *Fokker-Plank* equation.

Invariance

Invariance $\Rightarrow$ $$\int \mathbb{E}[F(\Phi_t^N(u))]\mathrm{d}\rho_N(u) = \int F(u)\mathrm{d}\rho_N(u).$$

We want to show

$$\int \mathbb{E}[F(\Phi_t(u))]\mathrm{d}\rho(u) = \int F(u)\mathrm{d}\rho(u).$$

Sanity check: is the flow defined for (at least) ρ-a.e. u?
Have to check $Z = S(t)(u, u_t) \in C_{t,x}$.
The typical (u, u_t) according to ρ is in $H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}$, so $Z \in H^{\frac{1}{2}-} \Rightarrow$ not enough.
However, like for ψ, $Z \in C_t C_x^{\frac{1}{2}-}$ ρ-a.s.. Moreover, $Z(P_N u) \to Z(u)$ in $C_t C_x^{\frac{1}{2}-}$ as $N \to \infty$, from which we can prove

$$\Phi_t^N(P_N u) \to \Phi_t(u) \text{ as } N \to \infty.$$

Invariance

ρ can be defined as the limit of ρ_N, in the sense that, if F bounded, $M > 0$,

$$(**) \qquad \int F(P_M u)\mathrm{d}\rho(u) = \lim_{N\to\infty} \int F(P_M u)\mathrm{d}\rho_N(u).$$

Take F bounded and Lipschitz. Then

$$\begin{aligned}
&\int \mathbb{E}[F(\Phi_t(u))]\mathrm{d}\rho(u) \\
&= \lim_M \int \mathbb{E}[F(\Phi_t(P_M u))]\mathrm{d}\rho(u) + \lim_M \int \mathbb{E}[F(\Phi_t(u)) - F(\Phi_t(P_M u))]\mathrm{d}\rho(u) \\
&\overset{(**)}{=} \lim_M \lim_N \int \mathbb{E}[F(\Phi_t(P_M u))]\mathrm{d}\rho_N(u) \\
&\overset{\text{Inv.}}{=} \lim_M \lim_N \int F(P_M u)\mathrm{d}\rho_N(u) + \lim_M \lim_N \int \mathbb{E}[F(\Phi_t(P_M u) - \Phi_t^N(P_M u))]\mathrm{d}\rho_N(u) \\
&\overset{(**)}{=} \lim_M \lim_N \int F(P_N P_M u)\mathrm{d}\rho_N(u) \overset{(**)}{=} \lim_M \int (F(P_M u)\mathrm{d}\rho(u) \\
&= \int F(u)\mathrm{d}\rho(u)
\end{aligned}$$

Ergodicity

Definition:

An invariant probability measure ρ is **ergodic** if one of the following equivalent conditions holds:

i) If a set S is invariant, i.e. $\Phi_t(S) \subset S$ $\mathbb{P}$-a.s. $\forall t$, then $\rho(0) = 0$ or $\rho(0) = 1$.

ii) If $\rho_1 \ll \rho$ invariant, then $\rho_1 = \rho$.

iii) $\nexists$ ρ_1, ρ_2 invariant s.t. $\rho_1 \perp \rho_2$ and $\rho_1, \rho_2 \ll \rho$

iv) If $\mathbb{E}[F(\Phi_t(u))] = F(u)$ for ρ-a.e. u, then F is constant ρ-a.e..

Heuristically, ergodicity means that we cannot split ρ into “smaller” invariant pieces.

Ergodicity

Geometric mixing

Let ρ be ergodic, $\bar{u} \in \text{supp}(\rho)$, and suppose that $\text{supp}(u)$ is not bounded. Then for every $\varepsilon > 0$, and for ρ almost every u, $\Phi_t(u)$ enters and exits B_ϵ infinitely many times a.s..

Birkhoff's Theorem

ρ is ergodic if and only if for every F bounded measurable,

$$\frac{1}{T}\int_0^T F(\Phi_t(u)) \to \int F(u)\mathrm{d}\rho(u)\mathrm{d}t$$

for ρ-a.e. u.

Main Result

Theorem: T. - in preparation

The Gibbs measure

$$\rho = \text{“}A^{-1} \exp(-H(u, u_t))\mathrm{d}u\mathrm{d}u_t\text{”}$$

is ergodic for the flow of (SDNLB).
Moreover, if the initial data $(u_0, u_1) \in H^2$,

$$\frac{1}{T}\int_0^T F(\Phi_t(u))\mathrm{d}t \to \int F(u)\mathrm{d}\rho(u) \;\; a.s.$$

Strong Feller Property

How to prove ergodicity?

Definition: Strong Feller Property

We say that Φ_t has the **Strong Feller Property** if $\mathbb{E}[F(\Phi_t(u))]$ is a continuous function of u for every F bounded and measurable.

Systems of SDEs: Hörmander condition $\Rightarrow$ Strong Feller.

Stochastic Navier Stokes: 2D: Da Prato-Debussche '02, 3D: Zhu-Zhu '15

SQE in 2d: Rockner-Zhu-Zhu '16, Tsatsoulis-Weber '16,

Hairer-Mattingly '17: very large class of parabolic stochastic equations, including KPZ, SQE in 3d.

Proposition

Suppose that ρ is invariant $\operatorname{supp}(\rho)$ is connected, and ρ satisfies the Strong Feller property. Then ρ is ergodic.
Moreover, if $\rho' \neq \rho$ is ergodic, $\operatorname{supp}(\rho') \cap \operatorname{supp}(\rho) = \varnothing$.

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Moreover, if $\rho' \neq \rho$ is ergodic, $\text{supp}(\rho') \cap \text{supp}(\rho) = \emptyset$.

Hairer - Mattingly '17: Suppose we can solve the control problem

$$\Phi_t(u+w,\xi) = \Phi_t(u,\xi + h(u,w,t,\xi)).$$

$$\begin{aligned}
\mathbb{E}[\Phi_t(u+w,\xi)] &= \int \Phi_t(u+w,\xi)\mathrm{d}\nu(\xi) \\
&= \int \Phi_t(u,\xi+h)\mathrm{d}\nu(\xi) \\
\text{Girsanov} \qquad &= \int \Phi_t(u,\xi)\mathcal{E}(h)\mathrm{d}\nu(\xi).
\end{aligned}$$

Since $\int \Phi_t(u,\xi)\mathrm{d}\nu(\xi) = \mathbb{E}[\Phi_t(u,\xi)]$, continuity in $u \approx \mathbb{E}(h) \to 1$ as $w \to 0$.

Problem: Girsanov $\Rightarrow h \in L^2$.

Recall that $\Phi_t(u,\xi) = \psi(\xi) + S(t)(u_0,u_1) + v(u)$, with $v(u) \in H^2$.
If we can solve $\Phi_t(u+w,\xi) = \Phi_t(u,\xi + h(u,w,t,\xi))$, with $h \in L^2_x$, then

$$\begin{cases} \Phi_t(u,\xi+h) = \psi(\xi) + \psi(h) + S(t)(u_0,u_1) + v(u) \\ \Phi_t(u+w,\xi) = \psi(\xi) + S(t)(u_0,u_1) + S(t)(w_0,w_1) + v(u+w) \end{cases}$$

If $h \in L^2$, then $\psi(h) \in H^2$. Therefore

$$S(t)(w_0,w_1) \in H^2 \Rightarrow (w_0,w_1) \in H^2 \times L^2.$$

But a generic difference between initial data according to ρ is in $H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-} \Rightarrow$ we **cannot** build such h.

Proposition

Φ_t is **not** Strong Feller in the $H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}$ topology.

However, if $(w_0, w_1) \in H^2 \times L^2$, there is still hope.

Proposition

If $(w_0, w_1) \in H^2 \times L^2$, we can find $h \in L^2$ s.t.

$$\Phi_t(u, \xi + h) = \Phi_t(u + w, \xi).$$

Moreover, Φ_t is Strong Feller in $H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}$ with the topology given by

$$d(u, u') = \|u - u'\|_{H^2 \times L^2} \vee 1.$$

However, this space (with this topology) does not satisfy any of the usual assumptions: it is disconnected, not separable, and $\mathrm{supp}(\rho) = \varnothing$.

Proposition

Let $\rho_1 \perp \rho_2$ be invariant. Then there exists S such that

$$\rho_1(S + (H^2 \times L^2)) = \rho_1(S) = 0, \qquad \rho_2(S + (H^2 \times L^2)) = \rho_1(S) = 1.$$

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Projection

The previous remarks suggest that we can throw away everything that belongs to $H^2 \times L^2$. Therefore, we consider the (algebraic) projection

$$\pi : (H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}) \to (H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}) \Big/ (H^2 \times L^2).$$

Recalling the decomposition $\Phi_t(u) = \psi(\xi) + Z(u) + v$, since $v \in H^2$,

$$\pi(\Phi_t(u)) = \pi(\psi(\xi) + Z(u) + v) = \pi(\psi(\xi) + Z(u)).$$

Therefore, if $L_t(u)$ is the flow of the *linear* equation

$$\text{(SBE)} \qquad \partial_t^2 u + \partial_t u + (1 + \Delta^2)u = \sqrt{2}\xi,$$

$\pi(\Phi_t(u)) = \pi(L_t(u))$. Since L_t is linear and maps $(H^2 \times L^2)$ into itself, there exists $\tilde{L}_t : (H^{\frac{1}{2}-} \times H^{-\frac{3}{2}-}) \Big/ (H^2 \times L^2) \circlearrowleft$ such that

$$\tilde{L}_t(\pi(u)) = \pi(L_t(u)) = \pi(\Phi_t(u)).$$

Linear Flow

Consider the flow L_t. Since $L_t u = \psi_t(\xi) + S(t)(u_0, u_1)$, and $S(t)$ has exponential decay as $t \to \infty$, then L_t admits a *unique* invariant measure μ, and therefore this measure will be ergodic.

Consider the measure $\pi_\sharp\mu$. This is be ergodic for $\tilde{L}$: if $\pi(S)$ is invariant for $\tilde{L}$, then $\pi^{-1}(\pi(S))$ will be invariant for L_t, and by ergodicity of L_t,

$$\pi_\sharp\mu(S) = \pi^{-1}(\pi(S)) = 0 \text{ or } 1.$$

Ergodicity

Consider now $\rho_1, \rho_2 \ll \rho$ invariant, $\rho_1 \perp \rho_2$. Take S such that

$$\rho_1(S + (H^2 \times L^2)) = 1, \;\; \rho_2(S + (H^2 \times L^2)) = 0.$$

Since $\rho_i \ll \rho \ll \mu$, $\pi_\sharp \rho_i \ll \pi_\sharp \mu$, and it will be invariant for $\pi \circ \Phi_t = \tilde{L}$.

$$\text{Ergodicity} \implies \pi_\sharp \rho_i = \pi_\sharp \mu.$$

Then

$$\begin{aligned} 1 &= \rho_1(S + (H^2 \times L^2)) = \pi_\sharp \rho_1(\pi(S)) = \pi_\sharp \mu(\pi(S)) \\ &= \pi_\sharp \rho_2(\pi(S)) = \rho_2(S + (H^2 \times L^2)) = 0, \end{aligned}$$

contradiction.