Asymptotics for the Hartree equation

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DISPERSIVE EQUATIONS WITH RANDOM INITIAL DATA
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Hamiltonian

We consider a system of N particles represented by a wave function ψ and evolving according to the energy :

$$\mathcal{E}_N = -\frac{1}{2}\sum_i \int \overline{\psi}\, \triangle_i\, \psi + \frac{1}{2}\sum_{i\neq j} w(x_i - x_j)\overline{\psi}\psi.$$

Here, we assume that the particles are fermions, which means that they satisfy Pauli's principle, in other words for all permutation σ

$$\psi(x_{\sigma(1)}, \dots x_{\sigma(n)}) = \varepsilon(\sigma)\psi(x_1, \dots x_n).$$

We take ψ to be equal to

$$\psi = \frac{1}{\sqrt{N!}}\sum_\sigma \varepsilon(\sigma)\prod_i u_{\sigma(i)}(x_i)$$

with u_i orthogonal and normalised (Slater determinant).

Hartree-Fock

We get the following energy :

$$\mathcal{E}_N = \mathcal{E}_{kin} + \mathcal{E}_{mf} + \mathcal{E}_{exch}$$

with

$$\begin{aligned}
\mathcal{E}_{kin} &= -\frac{1}{2}\sum_i u_i \bigtriangleup u_i \\
\mathcal{E}_{mf} &= \frac{1}{2}\sum_{i\neq j}\int w * |u_i|^2|u_j|^2 \\
\mathcal{E}_{exch} &= \frac{1}{2}\sum_{i\neq j}\int w * (\overline{u}_i u_j)\overline{u}_j u_i .
\end{aligned}$$

Hartree-Fock

By assuming $\mathcal{E}_{exch} \ll \mathcal{E}_N$ and $\mathcal{E}_{mf} \sim \frac{1}{2}\sum_{i,j}\int w * |u_i|^2|u_j|^2$, we get the equations

$$i\partial_t u_j = - \triangle u_j + w * (\sum_k |u_k|^2) u_j.$$

This equation was derived from many body quantum mechanics in a mean-field or semi-classical limit :

- Bardos Erdos Golse Mauser Yau '02
- Bardos Golse Gottlieb Mauser '03
- Elgart Erdos Schlein Yau '04
- Frohlich Knowles '11
- Benedikter Porta Schlein '13

Equation on operators

Take $\gamma = \sum_k |u_k\rangle\langle u_k|$ where $|u_k\rangle\langle u_k|$ is the projection over $\mathbb{C}u_k$. Note that $\mathrm{Tr}\,\gamma = N$ is the number of particles. It satisfies

$$i\partial_t\gamma = [-\triangle + w * \rho_\gamma, \gamma] \tag{1}$$

where $[\cdot,\cdot]$ is the commutator and ρ_γ is the diagonal of the integral kernel of γ, that is

$$\rho_\gamma(x) = \tilde{\gamma}(x,x)$$

with

$$(\gamma v)(x) = \int \tilde{\gamma}(x,y)v(y)dy.$$

Rk : Nonlinearity induced by ρ_γ.
Well-posed if $(-\triangle)^s\gamma$ is trace class (the regularity depends on the assumptions on w).

Stationary solutions

γ_f : Fourier multiplier by f.

$$\widehat{\gamma_f v}(\xi) = f(\xi)\hat{v}(\xi).$$

$\tilde{\gamma}_f(x,y) = \int d\xi f(\xi) e^{i\xi(x-y)}$
$\rho_{\gamma_f} = \int f(\xi) d\xi$.
γ_f commutes with $\triangle$ and $w * \rho_{\gamma_f}$ is a constant hence γ_f is a stationary solution. Rk : γ_f is not trace class ($\mathrm{Tr}\,(\gamma) = \int dx \rho_\gamma(x)$).
$Q = \gamma - \gamma_f$

$$i\partial_t Q = [-\triangle, Q] + [w * \rho_{Q+\gamma_f}, \gamma_f + Q]. \tag{2}$$

Global well-posedness

Lewin-Sabin : $f = 1_{[-\mu,\mu]}$, $d \geq 2$, $w \in L^1 \cap L^\infty$, defocusing, GWP in the energy space

Bose and Fermi gas at positive temperature : $d \in \{1, 2, 3\}$, $w \in L^1 \cap L^\infty$ (+ other assumptions depending on the dimension), defocusing, GWP in the entropy space.

Chen, Hong, Pavlovic : $d = 2, 3$, $f = 1_{[-\mu,\mu]}$, $w = \delta$, GWP in the energy space

Stability

Lewin-Sabin : $d = 2$, $w \in W^{1,1}$ + assumptions on f and w that are not smallness assumptions, scattering in Schatten $\frac{4}{3}$ for small data.

Chen, Hong, Pavlovic : $d \geq 3$, same assumptions on f, $w = w_+ * w_-$, $w_\pm \in L^{2d/(d+2)}$, $|\nabla|^{\pm\frac{1}{2}} w_\pm \in W^{\pm\alpha_0,1}$, ($\alpha_0$ depends on the regularity of the initial datum) + assumptions on f and w that are not smallness assumptions, scattering in Schatten $2d$ for small data.

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Model

Operators	Random Fields
$\gamma = \mathbb{E}(\|X\rangle\langle X\|)$	X
ρ_γ	$\mathbb{E}(\|X\|^2)$
$i\partial_t \gamma = [-\triangle + w * \rho_\gamma, \gamma]$	$i\partial_t X = -\triangle X + w * \mathbb{E}(\|X\|^2)X$
$\gamma_{\|f\|^2}$	$Y_f = \int f(k) e^{ikx - it(m+k^2)} dW(k)$

with $\mathbb{E}(dW(k)\overline{dW(l)}) = \delta(k-l)dkdl$ and $\mathbb{E}(|Y|^2) = \int |f|^2$ and $m = w * \int |f|^2$.
Rk : Law of Y_f is invariant by translations in both time and space.
$Z = X - Y_f$

$$i\partial_t Z = (m - \triangle)Z + w * (2\mathrm{Re}\mathbb{E}(\bar{Y}_f Z) + |Z|^2)(Y_f + Z)$$

GWP in the energy space $L^2_{\mathrm{prob}}, H^1(\mathbb{R}^3)$.

Retrieving the system

If one takes the probability space $\Omega = \{1, \dots, N\}$ with uniform probability, the random field X becomes

$$\sqrt{N}(u_1, \dots, u_N)$$

and

$$\mathbb{E}(|X|^2) = \sum_{k=1}^{N} |u_k|^2$$

and we retrieve the initial system.

GWP at the operator level

Existence :

$$\begin{array}{ccc} \gamma_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \gamma & \longleftarrow & X \end{array}$$

X_0 : Gaussian field with cov op γ_0

Uniqueness : γ_0 induces X_0.

γ_1, γ_2 two solutions with id $\gamma_0 \Rightarrow X_1, X_2$ two solutions with id X_0 and such that $\gamma_i = \mathbb{E}(|X_i\rangle\langle X_i|)$.

Continuity : Choice of appropriate metric space for γ.

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Equilibrium

The equation is for this section : $i\partial_t X = -\triangle X + \mathbb{E}(|X|^2)X$.

Write $Y_f = e^{-it} \int f(\xi)dW(\xi)$ with $\int |f(\xi)|^2 d\xi = 1$.

It solves $i\partial_t Y_f = -\triangle Y_f + \mathbb{E}(|Y_f|^2)Y_f$.
$X = Z + Y_f(t)$

$$i\partial_t Z = (1 - \triangle)Z + (\mathbb{E}(|Z|^2) + 2\mathrm{Re}\mathbb{E}(\bar{Y}_f Z))(Y_f + Z).$$

$Z_1 = \mathbb{E}(\bar{Y}_f Z)$, $Z_2 = Z - Z_1 Y_f$

$$\begin{cases} i\partial_t Z_1 = (1 - \triangle)Z_1 + 2\mathrm{Re}Z_1 + (\mathbb{E}(|Z_2|^2) + |Z_1|^2)(1 + Z_1) + 2\mathrm{Re}(Z_1)Z_1 \\ i\partial_t Z_2 = (1 - \triangle)Z_2 + (\mathbb{E}(|Z_2|^2) + |Z_1|^2 + 2\mathrm{Re}Z_1)Z_2 \end{cases}$$

Stability of e^{-it} for the Schrödinger equation

Gustafson-Nakanishi-Tsai :

$$i\partial_t u = (1 - \triangle)u + 2\mathrm{Re}u + |u|^2(1 + u) + 2\mathrm{Re}uu$$

scattering for small data of regularity $d/2 - 1$ for $d \geq 4$. + dim 3

Guo-Hani-Nakanishi : same equation, $d = 3$, scattering for small data of regularity of one more angular derivative than the energy space.

What we expect

d=4; GP with quadratic and cubic terms + Schrödinger with quadratic and cubic terms, should have scattering in the same way.

d=3 ; Problem comes from quadratic terms in the Schrödinger equation. Proof of relies on normal form. It is not obvious that it would work with the coupling.

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Equilibrium

$$Y = \frac{m}{2}(g_+ e_+ + g_- e_-)$$

with $e_\pm = e^{\pm i\xi_0 x - it(m+\xi_0^2)}$ and g_+, g_- two gaussians $\mathcal{N}(0,1)$ independent.
$\mathbb{E}(|Y|^2) = m$ and $i\partial_t Y = (m - \triangle)Y$.

Write $Z = g_+ e_+ z_+ + g_- e_- z_-$.

$$i\partial_t z_\pm = -\triangle z_\pm - 2i(\pm\xi_0)\cdot\nabla z_\pm + (|z_+|^2 + |z_-|^2 + 2\mathrm{Re}(z_+ + z_-))(\frac{m}{2} + z_\pm).$$

Linearised

$$i\partial_t z_\pm = - \triangle z_\pm - 2i(\pm\xi_0) \cdot \nabla z_\pm + m\mathrm{Re}(z_+ + z_-).$$

$$A = \begin{pmatrix} -2\xi_0 \cdot \nabla & -\triangle & 0 & 0 \\ \triangle - m & -2\xi_0 \cdot \nabla & -m & 0 \\ 0 & 0 & 2\xi_0 \cdot \nabla & -\triangle \\ -m & 0 & \triangle - m & 2\xi_0 \cdot \nabla \end{pmatrix}$$

Particular cases

$m = 0$ Dz

$$A = \mathrm{Diag}\,(-2\xi_0 \cdot \nabla + i\triangle; -2\xi_0 \cdot \nabla - i\triangle; 2\xi_0 \cdot \nabla + i\triangle; 2\xi_0 \cdot \nabla - i\triangle).$$

$\xi_0 = 0$ Dz

$$A = \mathrm{Diag}\,(i\sqrt{-\triangle\,(2m-\triangle)}; -i\sqrt{-\triangle\,(2m-\triangle)}; i\triangle; -i\triangle).$$

Otherwise, one always get spectral instability, as in : Let $\xi_0 \neq 0$ and $m \neq 0$. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that for all $\kappa \in \mathbb{S}^{d-1}$ and $r \in \mathbb{R}_+$; $|\kappa \cdot \xi_0| \leq \varepsilon_1$ and $r \leq \varepsilon_2$ implies that $\hat{A}(\xi)$ has real eigen values.

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Problem

We are back with $w \neq \delta$ and f smooth.

The perturbation Z satisfies the equation :

$$i\partial_t Z = (m - \triangle)Z + w * (2\mathrm{Re}\mathbb{E}(\bar{Y}_f Z) + \mathbb{E}(|Z|^2))(Y_f + Z)$$

with

$$Y_f = \int f(k) e^{ikx - it(m+k^2)} dW(k).$$

Result

Non-smallness assumption on f and w.

$$\|\hat{w}_-\|_{L^\infty} \lesssim \Big(\int_{\mathbb{R}^d} \frac{|h(x)|}{|x|^{d-2}} dx \Big)^{-1}$$

where h is the inverse Fourier transform of $|f|^2$.

$$\hat{w}(0)_+ < \varepsilon_f$$

where ε_f depends on the behaviour of f in 0.

Result

THEOREM [Collot, dS] Assume $|\xi|^{-1/2-\varepsilon}\hat{w} \in L^\infty$, and $w \in W^{d/2-1,1}$ and f of Schwartz class (simplification). Let X be the solution to

$$i\partial_t X = -\triangle X + w * \mathbb{E}(|X|^2)X$$

with initial datum $X_0 = Y_f(0) + Z_0$ with $Z_0 \in L^2_{\text{prob}}, H^{d/2-1}$, Z_0 independent from $Y_f(0)$ (simplification, there is a space) and small enough. Then, there exists $Z_{\pm\infty} \in L^2_{\text{prob}}, H^{d/2-1}$ such that

$$\|X(t) - Y_f(t) - S(t)Z_{\pm\infty}\|_{H^{d/2-1}}$$

goes to 0 when $t \to \pm\infty$.

Duhamel formulation

With $V = 2\mathrm{Re}\mathbb{E}(\bar{Y}_f Z) + \mathbb{E}(|Z|^2)$ and $S(t) = e^{-it(m-\triangle)}$:

$$Z(t) = S(t)Z_0 - i\int_0^t S(t-\tau)w * V(\tau)Y_f(\tau)d\tau - i\int_0^t S(t-\tau)V(\tau)Z(\tau)d\tau.$$

$$\begin{aligned} V(t) = {} & \mathbb{E}(|Z(t)|^2) + 2\mathrm{Re}\mathbb{E}(\overline{Y_f(t)}S(t)Z_0) \\ & - 2\mathrm{Re}\mathbb{E}\Big(i\overline{Y_f(t)}\int_0^t S(t-\tau)w * V(\tau)Y_f(\tau)d\tau\Big) \\ & - 2\mathrm{Re}\mathbb{E}\Big(i\overline{Y_f(t)}\int_0^t S(t-\tau)w * V(\tau)Z(\tau)d\tau\Big). \end{aligned} \quad (3)$$

Discussion on spaces

Space for V

$$\Theta_V = L^2, L^2 \cap L^{1+d/2}, L^{1+d/2}$$

cannot have $V \in L^p, L^p$ with $p < 2$, or derivatives in V, implies loss of derivatives in $w : s = \frac{d}{2} - 1$.
Space for Z

$$\Theta = L^4, L^4 \cap L^{d+2}, L^{d+2} \cap L^{2(2+d)/d}, W^{s,2(2+d)/d} \cap C(\mathbb{R}, H^s)$$

$$\|S(t)g\|_\Theta \lesssim \|g\|_{H^s}.$$

Linear term

The assumptions of “non smallness” on f are the same assumptions on f as in Lewin-Sabin, or Chen-Hong-Pavlovic.

Linear term :

$$-2\mathrm{Re}\mathbb{E}\Big(i\overline{Y_f(t)}\int_0^t S(t-\tau)V(\tau)Y_f(\tau)d\tau\Big) = -L(V)$$

Problem : Proving that $1+L$ is continuously invertible without a smallness assumption on Y_f.

$$\widehat{L(V)} = -2\int_0^t d\tau \sin((t-\tau)\xi^2)h(2(t-\tau)\xi)\hat{w}(\xi)\hat{V}(\tau,\xi).$$

Linear term

$$L(V) = K *_{t,x} V.$$

We have

$$\|L(V)\|_{\Theta_V} \lesssim \|\mathcal{F}_{t,x}(L(V))\|_{L^2, L^2 \cap L^{(d+2)/d, (d+2)/d}}.$$

Hence it suffices to bound $\mathcal{F}_{t,x}(K)$ in $L^2, L^2 \cap L^p, L^p$ with $p = 2\frac{d+2}{d-2}$.
If it is small enough we get that $1 + L$ is invertible.
The sine helps for small ξ otherwise one may use the decay of h at infinity.

Additional difficulties

Bounding

$$I = -i \int_0^t S(t-\tau)V(\tau)Y_f(\tau)d\tau$$

in the space for Z. Requires $\hat{w}(k) = O(|k|^{1/2+})$.
Indeed,

$$\mathbb{E}(|I|^2) = \int d\eta |f(\eta)|^2 \Big| \int_0^t S_\eta(t-\tau)(w * V(\tau))d\tau \Big|^2$$

where

$$S_\eta(t) = S_\eta(t) = e^{-it(-\triangle - 2i\eta\cdot\nabla)}.$$

It reduces using Strichartz inequalities to bound

$$\int_0^\infty dt_1 \int_{-t_1}^\infty dt \int d\xi e^{it\xi^2} h(-2t\xi)(1+\xi^2)^s |\hat{w}(\xi)|^2 \hat{V}(t_1,\xi)\overline{\hat{V}(t_1+t,\xi)}.$$

We would use the L^2, L^2 norm of V. The issue is the lack of sine. That is where we use the hypothesis on the behaviour at 0 of $\hat{w}$.

Additional difficulties, II

Bounding

$$-2\mathrm{Re}\mathbb{E}\Big(i\overline{Y_f(t)}\int_0^t S(t-\tau)V(\tau)Z(\tau)d\tau\Big).$$

by iteration.

Calling $W_V(g) = \int_0^t S(t-\tau)V(\tau)g(\tau)d\tau$, this means bounding

$$-2\mathrm{Re}\mathbb{E}\Big(i\overline{Y_f(t)}W_V^n(Y)d\tau\Big).$$

We can compute this term explicitely and use the same technique as in the linear term and I.

Thank you for your attention !