

L -functions

Andrew Booker

June 2018

What is an L -function?

One possible answer: the Dirichlet series associated to automorphic representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$.

Problems with this answer: The words in it are not easy to define, and there are many things we don't know how to prove about them (Langlands' conjectures).

Selberg's idea (1989): Try an abstract approach, defining L -functions axiomatically, and think about the whole lot of them at once. Perhaps we can see how they are related without recourse to automorphic forms.

The Selberg class

Let $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. Suppose:

- 1 Analytic continuation: $(s-1)^m F(s)$ continues to an entire function of finite order for some integer $m \geq 0$;
- 2 Ramanujan hypothesis: $a_n \ll_{\varepsilon} n^{\varepsilon}$;
- 3 Functional equation: for certain numbers $Q > 0$, $\omega \in \mathbb{C}^{\times}$, $\lambda_j > 0$ and $\mu_j \in \mathbb{C}$ with non-negative real part, if

$$\gamma(s) = \omega Q^s \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j) \quad \text{and} \quad \Phi(s) = \gamma(s)F(s),$$

then $\Phi(s) = \overline{\Phi(1-\bar{s})}$.

- 4 Euler product: $F(s) = \prod_p F_p(s)$ for $\Re(s) > 1$, where $\log F_p(s) = \sum_{n=1}^{\infty} \frac{b_{p^n}}{p^{ns}}$ satisfies $b_{p^n} = O(p^{n\theta})$ for some $\theta < \frac{1}{2}$.

The set of all such F is called the *Selberg class*, denoted S .

Selberg's conjectures

- Unique factorization: The functions in S can be written uniquely as products of primitive elements.
- Orthogonality: If $F, G \in S$ are primitive elements with Dirichlet coefficients a_F and a_G , then

$$\sum_{p \leq x} \frac{a_F(p) \overline{a_G(p)} - \delta_{F=G}}{p} = O_{F,G}(1).$$

- Stability under twists: If $F \in S$ and $\chi \pmod{q}$ is a primitive Dirichlet character then there is an $F^\chi \in S$ with $a_{F^\chi}(n) = a_F(n)\chi(n)$ for n co-prime to q .
- Riemann hypothesis: All zeros of Φ have real part $\frac{1}{2}$.
- (Later) Degree conjecture: $d = 2 \sum_{j=1}^k \lambda_j$ is an integer.

- The elements of S of degree $d \in [0, 2)$ have been completely classified. They are the constant function 1 (of degree 0) and the shifted Dirichlet L -functions $L(s + it, \chi)$ for primitive characters χ (of degree 1).
- With the possible exception of the Riemann hypothesis, all of Selberg's conjectures are (trivially) true for these L -functions.
- The general expectation is that all elements of the Selberg class are automorphic L -functions.
- This is completely open for degrees $d \geq 2$.

Some problems with the axiomatic approach

- It is not obvious which properties of L -functions should be taken as axioms and which are theorems to be derived from the axioms.
- In particular, Selberg's axioms do not correspond well with the properties of the L -functions that we know about, i.e. those associated to automorphic forms. For instance, the Ramanujan conjecture is unproven, but orthogonality of the coefficients is essentially known (Rankin–Selberg).
- The definition makes some *ad hoc* conventions. For instance, the Γ -factors of all known L -functions can be expressed in terms of $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$. Selberg generalizes this to allow $\Gamma(\lambda s)$ for arbitrary $\lambda > 0$, but does not allow the analogue for the finite Euler factors $(1/(1 - p^{-2\lambda s}))$.
- In my opinion, this has led to a contrived sense of generality.

HOW STANDARDS PROLIFERATE:

(SEE: A/C CHARGERS, CHARACTER ENCODINGS, INSTANT MESSAGING, ETC.)



Vague idea #1

Change language completely. Instead of speaking directly about L -functions, whose definition we cannot agree on, try to characterize them in terms of their *explicit formulae*. Perhaps this will enable a less *ad hoc* formulation with useful applications.

The explicit formula relates the coefficients of an L -function to its zeros via an identity of distributions. For example, if $\chi \pmod{q}$ is an even primitive Dirichlet character and $g : \mathbb{R} \rightarrow \mathbb{C}$ is a sufficiently nice test function (e.g. smooth of compact support) with Fourier transform $h(z) = \int_{\mathbb{R}} g(x)e^{izx} dx$ satisfying $h(\mathbb{R}) \subseteq \mathbb{R}$, then

$$\sum_{z \in \mathbb{C}} m(z)h(z) = 2\Re \left[\int_0^\infty (g(0) - g(x)) \frac{e^{-x/2}}{1 - e^{-2x}} dx \right. \\ \left. + \frac{1}{2} \left(\log \frac{q}{8\pi} - \gamma - \frac{\pi}{2} \right) g(0) - \sum_{n=2}^\infty \frac{\Lambda(n)\chi(n)}{\sqrt{n}} g(\log n) \right],$$

where $m(z) = \text{ord}_{s=\frac{1}{2}+iz} \Gamma_{\mathbb{R}}(s)L(s, \chi)$.

Vague idea #1 (continued)

- The Γ -factor of $L(s, \chi)$ appears in the formula (in a transformed guise) as the integral kernel $\frac{e^{-x/2}}{1-e^{-2x}}$.
- A nice feature of expressing things this way is that it's immediately clear how to generalize—simply replace $\frac{e^{-x/2}}{1-e^{-2x}}$ by a more general function!
- There are a couple subtleties:
 - The kernel should have a first-order singularity at 0. This is present for all automorphic L -functions, with residue indicating the *degree*.
 - The Γ -factor also contributes to the point mass at 0 (i.e. the $g(0)$ term), and this must also be generalized. This turns out to yield a natural notion of *analytic conductor*.

Vague idea #2

For Dirichlet series with functional equation, the existence of an Euler product is morally equivalent to non-vanishing in the *region of absolute convergence*. Therefore, in the context of the explicit formula, one should be able to *dispense with the Euler product*.

(This should not be taken literally, since there are easy counterexamples, but they can be eliminated with a bit more precise statement.)

By serendipity, Frank Thorne asked me a related question around the time that I started daydreaming about this, and happened to be visiting Kyoto at the same time in 2013. We set out to prove:

Theorem (B.–Thorne, 2013)

Let $f \in S_k(\Gamma_1(N))$ be a holomorphic cuspform of arbitrary weight and level. If the associated complete L-function

$\Lambda_f(s) = \int_0^\infty f(iy)y^{s-1} dy$ does not vanish for $\Re(s) > \frac{k+1}{2}$ then f is an eigenfunction of the Hecke operators T_p for all primes $p \nmid N$.

Vague idea #2 (continued)

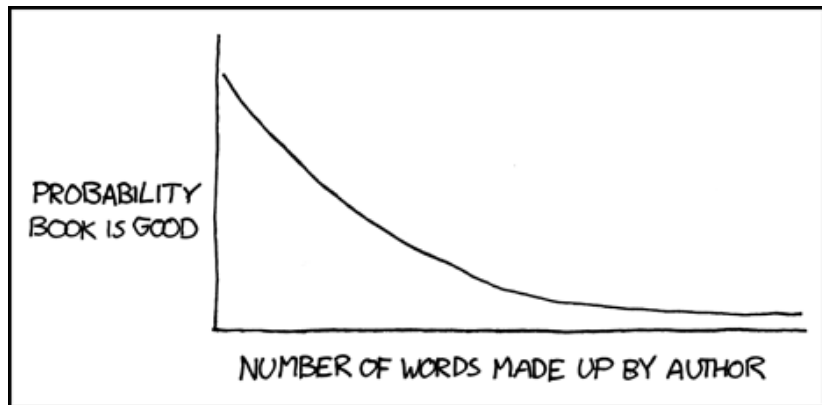
Theorem (B.–Thorne, 2013; Righetti, 2014)

Fix a positive integer n . For $j = 1, \dots, n$, let r_j be a positive integer and π_j a unitary cuspidal automorphic representation of $\mathrm{GL}_{r_j}(\mathbb{A}_{\mathbb{Q}})$ with L -series $L(s, \pi_j) = \sum_{m=1}^{\infty} \lambda_j(m) m^{-s}$. Assume that the π_j satisfy the generalized Ramanujan conjecture at all finite places (in particular, $|\lambda_j(p)| \leq r_j$ for all primes p) and are pairwise non-isomorphic.

Let $R = \left\{ \sum_{m=1}^M \frac{a_m}{m^s} : M \in \mathbb{Z}_{\geq 0}, (a_1, \dots, a_M) \in \mathbb{C}^M \right\}$ denote the ring of finite Dirichlet series, and let $P \in R[x_1, \dots, x_n]$ be a polynomial with coefficients in R .

Then either $P(L(s, \pi_1), \dots, L(s, \pi_n))$ has a zero with real part > 1 or $P = D(s)x_1^{d_1} \cdots x_n^{d_n}$ for some $D \in R$, $d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$.

The key point is precisely Selberg's orthogonality conjecture, which is known (in a slightly weaker form) for automorphic L -functions, by the Rankin–Selberg method.



"THE ELDERS, OR *FRA'AS*, GUARDED THE *FARMLINGS* (CHILDREN) WITH THEIR *KRYTOSES*, WHICH ARE LIKE SWORDS BUT *AWESOMER...*"

Definition

An L -datum is a triple $F = (f, K, m)$, where $f : \mathbb{Z}_{>0} \rightarrow \mathbb{C}$, $K : \mathbb{R}_{>0} \rightarrow \mathbb{C}$ and $m : \mathbb{C} \rightarrow \mathbb{R}$ are functions satisfying:

- (A1) $f(1) \in \mathbb{R}$, $f(n) \log^k n \ll_k 1$ for all $k > 0$, and $\sum_{n \leq x} |f(n)|^2 \ll_\varepsilon x^\varepsilon$ for all $\varepsilon > 0$;
- (A2) $xK(x)$ extends to a Schwartz function on \mathbb{R} , and $\lim_{x \rightarrow 0^+} xK(x) \in \mathbb{R}$;
- (A3) $\text{supp}(m) = \{z \in \mathbb{C} : m(z) \neq 0\}$ is discrete and contained in a horizontal strip $\{z \in \mathbb{C} : |\Im(z)| \leq y\}$ for some $y \geq 0$, $\sum_{\substack{z \in \text{supp}(m) \\ |\Re(z)| \leq T}} |m(z)| \ll 1 + T^A$ for some $A \geq 0$, and $\#\{z \in \text{supp}(m) : m(z) \notin \mathbb{Z}\} < \infty$;
- (A4) for every smooth function $g : \mathbb{R} \rightarrow \mathbb{C}$ of compact support and Fourier transform $h(z) = \int_{\mathbb{R}} g(x) e^{ixz} dx$ satisfying $h(\mathbb{R}) \subseteq \mathbb{R}$,

$$\sum_{z \in \text{supp}(m)} m(z) h(z) = 2\Re \left[\int_0^\infty K(x) (g(0) - g(x)) dx - \sum_{n=1}^\infty f(n) g(\log n) \right].$$

Definition (continued)

Given an L -datum $F = (f, K, m)$, we associate an L -function $L_F(s)$ defined by

$$L_F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s} = \exp\left(\sum_{n=2}^{\infty} \frac{f(n)}{\log n} n^{\frac{1}{2}-s}\right) \quad \text{for } \Re(s) > 1;$$

we call $d_F = 2 \lim_{x \rightarrow 0^+} xK(x)$ the *degree* of F and $Q_F = e^{-2f(1)}$ its *analytic conductor*, and we say that F is *positive* if there are at most finitely many $z \in \mathbb{C}$ with $m(z) < 0$.

Let \mathcal{L} denote the set of all L -data and $\mathcal{L}^+ \subseteq \mathcal{L}$ the subset of positive elements. Note that \mathcal{L} is a group with respect to addition, with identity element $(0, 0, 0)$, and \mathcal{L}^+ is a monoid. For any $d \in \mathbb{R}$, let $\mathcal{L}_d = \{F \in \mathcal{L} : d_F = d\}$ and $\mathcal{L}_d^+ = \mathcal{L}_d \cap \mathcal{L}^+$.

Examples

If $L(s) = \exp(\sum_{n=2}^{\infty} b(n)n^{-s})$ is an element of the Selberg class with complete L -function $\Phi(s) = \omega Q^s \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j) \cdot L(s)$, then there is an L -datum $F = (f, K, m) \in \mathcal{L}^+$ satisfying $d_F = 2 \sum_{j=1}^k \lambda_j$, $L_F(s) = L(s)$,

$$f(n) = \begin{cases} -\log Q - \Re \sum_{j=1}^k \lambda_j \frac{\Gamma'}{\Gamma}(\frac{\lambda_j}{2} + \mu_j) & \text{if } n = 1, \\ \frac{b(n) \log n}{\sqrt{n}} & \text{if } n > 1, \end{cases}$$

$$K(x) = \sum_{j=1}^k \frac{e^{-(\frac{1}{2} + \frac{\mu_j}{\lambda_j})x}}{1 - e^{-\frac{x}{\lambda_j}}}, \quad \text{and} \quad m(z) = \text{ord}_{s=\frac{1}{2}+iz} \Phi(s).$$

In particular, the estimate $\sum_{n \leq x} |f(n)|^2 \ll_{\varepsilon} x^{\varepsilon}$ follows from the Ramanujan hypothesis together with the bound $b(n) \ll n^{\theta}$.

Examples (continued)

If π is a unitary cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$ with conductor q , $L(s, \pi_{\infty}) = \prod_{j=1}^d \Gamma_{\mathbb{R}}(s + \mu_j)$, $-\frac{L'}{L}(s, \pi) = \sum_{n=2}^{\infty} c_n n^{-s}$ and $\Lambda(s, \pi) = L(s, \pi_{\infty})L(s, \pi)$, then there is an L -datum $F = (f, K, m) \in \mathcal{L}_d^+$ satisfying $L_F(s) = L(s, \pi)$,

$$f(n) = \begin{cases} -\frac{1}{2} \log q - \Re \sum_{j=1}^d \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\frac{1}{2} + \mu_j) & \text{if } n = 1, \\ \frac{c_n}{\sqrt{n}} & \text{if } n > 1, \end{cases}$$

$$K(x) = \sum_{j=1}^d \frac{e^{-(\frac{1}{2} + \mu_j)x}}{1 - e^{-2x}}, \quad \text{and} \quad m(z) = \mathrm{ord}_{s=\frac{1}{2}+iz} \Lambda(s, \pi).$$

In this case, the estimate $\sum_{n \leq x} |f(n)|^2 \ll \log^2 x$ for $x \geq 2$ follows from the Rankin–Selberg method, and the other conditions on f and K follow from partial results toward the Ramanujan conjecture.

Examples (continued)

- If $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_d(\mathbb{C})$ is an Artin representation then there is an L -datum $F = (f, K, m) \in \mathcal{L}_d$ with $L_F(s) = L(s, \rho)$, and f, K, m defined similarly to the case of automorphic L -functions. The Artin conjecture asserts that F is positive.
- Similarly, if E is an elliptic curve defined over \mathbb{Q} , then the symmetric power L -functions $L(s, \text{Sym}^k E)$ give rise to L -data.

Main results

The map $F \mapsto L_F$ defines a homomorphism from \mathcal{L} to the multiplicative group of non-vanishing holomorphic functions on $\{s \in \mathbb{C} : \Re(s) > 1\}$.

The first result shows that this map is injective, i.e. each L -datum is determined by its L -function, in the following strong sense.

Theorem (Multiplicity one)

For $F = (f, K, m) \in \mathcal{L}$, the following are equivalent:

- (i) $F = (0, 0, 0)$;
- (ii) $\sum_{n=2}^{\infty} \frac{|f(n)|}{\log n} < \infty$;
- (iii) $\sum_{n=1}^{\infty} \frac{|a_F(n)|}{\sqrt{n}} < \infty$;
- (iv) $L_F(s)$ is a ratio of Dirichlet polynomials;
- (v) $\sum_{\substack{z \in \text{supp}(m) \\ |\Re(z)| \leq T}} |m(z)| = o(T)$.

Main results (continued)

Second, the classification of the degree $d < \frac{5}{3}$ elements of the Selberg class, begun by Conrey–Ghosh and continued and refined by Kaczorowski–Perelli and Soundararajan, can be adapted to our setting.

Theorem (Converse theorem)

Let $F \in \mathcal{L}_d^+$ for some $d < \frac{5}{3}$. Then either $d = 0$ and $L_F(s) = 1$, or $d = 1$ and there is a primitive Dirichlet character χ and $t \in \mathbb{R}$ such that $L_F(s) = L(s + it, \chi)$.

Tom Oliver has since extended this to $d < 2$.

Corollary

Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$. Then its complete L -function $\Lambda(s, \pi)$ has infinitely many zeros of odd order.

Proof.

Let $F = (f, K, m) \in \mathcal{L}_3^+$ be the L -datum associated to π . If $\Lambda(s, \pi)$ has at most finitely many zeros of odd order then $m(z)$ is an even integer for all but at most finitely many z , and thus $\frac{1}{2}F \in \mathcal{L}_{3/2}^+$, in contradiction to the converse theorem. □

Corollary

For $j = 1, 2$, let π_j be a unitary cuspidal automorphic representation of $\mathrm{GL}_{d_j}(\mathbb{A}_{\mathbb{Q}})$ with complete L -function $\Lambda(s, \pi_j)$. If $d_2 - d_1 \leq 1$ and $\pi_1 \not\cong \pi_2$ then $\Lambda(s, \pi_2)/\Lambda(s, \pi_1)$ has infinitely many poles.

Proof.

Let $F \in \mathcal{L}$ be the L -datum with L -function $L_F(s) = L(s, \pi_2)/L(s, \pi_1)$, so that $d_F \leq 1$. If $\Lambda(s, \pi_2)/\Lambda(s, \pi_1)$ has at most finitely many poles then F is positive, so by the converse theorem, either $L_F(s) = 1$ or $L_F(s) = L(s + it, \chi)$ for some primitive Dirichlet character χ and $t \in \mathbb{R}$. However, neither of these is possible since $\pi_1 \not\cong \pi_2$ and π_2 is cuspidal. \square

Where do we go from here? (vague idea #3)

It would be good to find a formulation that incorporates twists (at least character twists). More elaborately:

- If \mathcal{L}^{aut} denotes the subgroup of \mathcal{L} generated by the L -data associated to unitary cuspidal automorphic representations of $\text{GL}_d(\mathbb{A}_{\mathbb{Q}})$ for all d , then \mathcal{L}^{aut} is not only a group, but (conjecturally) has the additional structure of a commutative ring, with the product corresponding to the tensor product of representations.
- The approach to classifying the elements of the Selberg class taken so far purposefully ignores most of this structure and relies essentially on Fourier analysis, which amounts to considering twists by n^{-it} , i.e. multiplication (in the above sense) by $F \in \mathcal{L}$ with $L_F(s) = \zeta(s + it)$.
- Such F are units in \mathcal{L}^{aut} , as are the L -data corresponding to $L(s + it, \chi)$ for primitive Dirichlet characters χ .
- Perhaps it would be more natural to build stability under twist by all units into the definition.

Vague idea #3 (continued)

Another reason to believe in this: the converse theorem sage Piatetski-Shapiro conjectured that analytic data from character twists should be enough to distinguish the automorphic representations among all irreducible admissible representations. So there is at least some hope of eventually classifying everything (or at least everything of integral degree) this way.

However, there is a hidden subtlety:

- In all known versions of the converse theorem for degree at least 2 (beginning with Weil), knowledge of the relationship between the root numbers and conductors of a given L -series and its twists is essential in the proof.
- The current definition of L -datum does not even mention the root number, and as our results demonstrate, it plays no role in the classification of low-degree elements of \mathcal{L}^+ .

So we must first try to clarify the role of the root number in the converse theorem.

The Euler product (or a weakening of it such as axiom (A1)) seems important for characterizing automorphic representations with minimal analytic data. A few data points:

- Weil's converse theorem for classical modular forms uses many twists but does not require an Euler product.
- It has been conjectured that a single functional equation (i.e. no twists) suffices, *assuming an Euler product*.
- However, there are examples of Dirichlet series with analytic continuation and modular-form-type functional equations that are not modular. Thus, one cannot eliminate *both* the twists and the Euler product.
- There is an example of a Shintani zeta-function which, *together with its twists by Dirichlet characters*, has the expected analytic properties of a degree 4 L -function, but is certainly not one.

It is far from clear why the Euler product helps, or how to make use of it in the converse theorem. This should be clarified.

- 1 Prove a converse theorem for classical holomorphic modular forms, assuming that all character twists satisfy the expected analytic properties, but *without knowledge of the root number*.
- 2 Prove a converse theorem for automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$, assuming axiom (A1) and that all character twists have the expected analytic properties, but *without requiring an Euler product*.

A. R. Booker, [A converse theorem without root numbers](#),
arXiv:1703.01834.

S. Bettin, J. W. Bober, A. R. Booker, B. Conrey, M. Lee,
G. Molteni, T. Oliver, D. J. Platt and R. S. Steiner,
[A conjectural extension of Hecke's converse theorem](#),
The Ramanujan Journal, 2017.

A. R. Booker and M. Krishnamurthy,
[A converse theorem for \$GL\(n\)\$](#) ,
Advances in Mathematics, 2016.

Main ideas of the proofs

- The statement “ $L(s)$ has analytic continuation and a functional equation” can be recast entirely in terms of the existence of an explicit-formula-type distributional identity (axiom (A4)).
- More precisely, one shows that there is a unique \mathbb{R}^\times function $\gamma_F(s)$ with properties reminiscent of the Γ -factors of automorphic L -functions (analytic for $\Re(s) \geq \frac{1}{2}$ + asymptotic expansion akin to Stirling’s formula), such that $\gamma_F(s)L_F(s)$ has meromorphic continuation and satisfies the functional equation $\gamma_F(s)L_F(s) = \overline{\gamma_F(1-\bar{s})L_F(1-\bar{s})}$.
- This plus the other technical conditions implies the multiplicity one result.
- For the classification, one studies the exponential sum $S_F(z) = \sum_{n=1}^{\infty} a_F(n)e^{2\pi inz}$ introduced by Conrey and Ghosh. The proof breaks into three cases, with very different behaviour: $d < 1$, $d = 1$ and $d \in (1, 2)$.

Sketch of the proof for $d = 1$

- Begin with the Mellin transform identity

$$S_F(z) = \frac{1}{2\pi i} \int_{\Re(s)=2} (2\pi)^{-s} \Gamma(s) L_F(s) (-iz)^{-s} ds.$$

- Apply the functional equation

$\gamma_F(s) L_F(s) = \overline{\gamma_F(1-\bar{s}) L_F(1-\bar{s})}$ and asymptotic expansion of $\gamma_F(s)$ to arrive at the transformed identity

$$S_F(z) = \frac{c}{2\pi i} \int_{\Re(s)=\frac{1}{2}+2\ell-\mu} \frac{(-iAz)^{s-1}}{\cos \frac{\pi}{2} \left(s - \frac{1}{2} + \mu\right)} \overline{L_F(\bar{s})} (1 + O(|s|^{-1})) ds + O(\Im(z)^{-\varepsilon})$$

for certain constants $c \in \mathbb{C}^\times$, $\mu \in \mathbb{R}$, $A \in \mathbb{R}_{>0}$, $\ell \in \mathbb{Z}_{>0}$.

- Expand the Dirichlet series for $L_F(s)$ and compute the integral (it's just the Fourier transform of sech), leading to

$$S_F(z) = O(\Im(z)^{-\varepsilon}) - \frac{2ic(-1)^\ell}{\pi} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n} \frac{(-iAz/n)^{2\ell-\frac{1}{2}-\mu}}{\frac{Az}{n} - \frac{n}{Az}}.$$

Sketch of the proof for $d = 1$ (continued)

- Now set $z = -(x - iy)/A$ and take $y \rightarrow 0^+$ to get

$$S_F(z) = \frac{ce^{i\frac{\pi}{2}(\frac{1}{2}-\mu)}}{\pi y} a_F(x) + O(y^{-\varepsilon}).$$

(Here $a_F(x)$ is understood to be 0 when x is not an integer!)

- The invariance of $S_F(z)$ under $z \mapsto z + 1$ implies that $a_F(n)$ is periodic with period A . In particular, A is an integer.
- The remaining technical conditions (in particular non-vanishing outside the critical strip) imply that $L_F(s)$ is the L -function of a primitive Dirichlet character of conductor dividing A .